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BUCKLING NEAR A HOLE IN AN
INFINITE PLATE UNDER TENSION

DAVID ANDREW PELLETT

BUCKLING NEAR A HOLE IN AN INFINITE PLATE
UNDER TENSION

by

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B.S.E.E., Purdue University, 1961



Submitted in partial fulfillment
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ABSTRACT

A study is made of the circumstances under which there will be incipient buckling in an infinite plate containing a circular hole and subjected to uniform uniaxial tension at a great distance from the hole. This buckling is a consequence of the fact that near the hole and in line with the direction of the applied tensile loading, there are areas in which both principal membrane stresses are compressive. Values of critical ~~tensile~~ stress are given, together with expressions for the corresponding lateral deflection patterns. The results are in the form

$$S_{cr} = KE(t/a)^2$$

where E is Young's modulus of elasticity, t is plate thickness, a is hole radius, and K is a buckling coefficient having the following values for the first four modes: 7.65, 7.80, 11.7, 12.3. A comparison is made with some experimental determinations of Danis for plates of finite width.

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NOMENCLATURE

a	Hole radius
A_{NM}	A set of unknown coefficients in a series describing plate deflection.
$\underline{A}$	An unknown eigenvector
A	An integer index used in Appendix C
a(J,L)	A trigonometric product used as an intermediate function in Tables 5 and 6
b	The half-width of an infinite strip in convenient units
b(J,L)	A trigonometric product used as an intermediate function in Tables 5 and 6
$\underline{\underline{B}}$	A coefficient matrix in Appendix D
B	An integer index in Appendix C
C	An integer index in Appendix C
d	Hole diameter
D	An integer index in Appendix C
D	Flexural rigidity of the plate. = $Eh^3/12 (1 - \nu^2)$
E	Young's Modulus
$E_1, E_2, \text{etc.}$	Terms in the membrane energy integral
f	Functions of I, J, k, and L used in Table 5
g	Functions of I, J, K, and L used in Table 6
$G_1, G_2, \text{etc.}$	Terms in the bending energy integral
h	Plate thickness in units of hole radii
$\underline{\underline{I}}$	Identity matrix
I	A dummy integer index for the index N
I'	A single subscript representing the double subscript IJ
$\bar{I}$	The maximum value of I'
J	A dummy integer index for the index M

NOMENCLATURE (continued)

J'	A single subscript representing the double subscript KL
$\bar{J}$	The maximum value of J'
K	A dummy integer index for the index N
K	Buckling coefficient based on hole radius
L	A dummy integer index for the index M
M	An integer index
M_r	Bending moment per unit length
M_{rt}	Twisting moment per unit length
N	An integer index
N_x, N_y	Membrane forces per unit length, $= \sigma_x h$, etc.
$P_{IJKL}, P_{I'J'}$	A term in the quadruple series that expresses the bending energy integral
$\underline{Q}$	A matrix of the eigenvectors of $\underline{I}$
$Q_{IJKL}, Q_{I'J'}$	A term in the quadruple series that expresses the membrane energy integral
r	Distance from the hole center in units of hole radii; also a coordinate of the cylindrical system
$R_{IJKL}, R_{I'J'}$	A term in the quadruple series that expresses the membrane energy integral
$\underline{R}$	The matrix of elements R_{IJKL}
s_1, s_2, s_3	Functions of N, M and ν in Appendix A.
S	Externally applied stress
S_{cr}	The value of S at which incipient buckling occurs
t_1, t_2, t_3	Functions of N, M and ν in Appendix A.
t	Plate thickness
$\underline{T}$	The matrix of elements T_{IJKL}
$T_{IJKL}, T_{I'J'}$	A term in the quadruple series that expresses the bending energy integral
u	Component of displacement in the x direction

NOMENCLATURE (continued)

U_1, U_2	Membrane energy in the plate
U_3	Bending energy in the plate
v	Component of displacement in the y direction
W	Plate deflection in units of hole radii
$W_r, W_{rr}, \text{etc.}$	Derivatives of plate deflection with respect to the subscripts
x^*	A compacted notation of X_{NM} in Table 5 with I and J replacing N and M respectively
x	A compacted notation for X_{NM} in Table 5 with K and L replacing N and M respectively; also a coordinate in the Cartesian system
X_{NM}	An unknown coefficient which is used to satisfy the boundary conditions for each term in N or M
<u>X</u>	An unknown eigenvector in Appendix D
y^*	A compacted notation for Y_{NM} in Table 5 with I and J replacing N and M respectively
y	A compacted notation for Y_{NM} in Table 5 with K and L replacing N and M respectively; also a coordinate of the Cartesian system; also a scalar parameter in an eigenvalue problem in Appendix D
Y_{NM}	An unknown coefficient which is used to satisfy the boundary conditions for each term in N or M
α	The number of terms taken in N in the series for W
β	The number of terms taken in M in the series for W
λ	A scalar parameter in eigenvalue problems
$\sigma_x, \sigma_y, \sigma_r, \sigma_\theta$	Normal stress
$\tau_{r\theta}, \tau_{xy}$	Shear stress
θ	An angular coordinate in the cylindrical system
ν	Poisson's ratio ($\nu = .3$ is used throughout)

OTHER NOTATION

A double underline indicates a square matrix. (A)

A single underline indicates a column vector. (A)

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1. INTRODUCTION

The problem of determining the stress situation near a hole in a plate under tension (see Figure 1) has a well known solution which is of great practical importance.

The original work on this problem was done by Kirsch (19), who solved the problem of the stress distribution in an infinite plate pierced with a round hole and subject to uniaxial tension at a great distance from the hole. Howland (17) treated a similar problem with an infinite strip of finite width. Savin (5) and Hogan (15) have compiled solutions produced by other authors for variations in hole geometry, as square, triangular, etc., for both finite and infinite plates. Lur'e (3) has solved the problem with a hole in a shell of uniform curvature.

An important result obtained from these solutions is that the stresses near the hole become much greater than those imposed upon the plate at a distance from the hole. This has given rise to the wide publication of many stress concentration factors which are important in design work and which relate the maximum stress to the nominal stress at a distance from the hole.

Another feature of the stress situation, which is not so widely recognized, is that certain areas of the plate are under uniaxial or biaxial compression,* although tension is imposed externally. These areas of compression, given the proper conditions of plate thickness

* These terms will be employed herein to mean that one or both principal stresses are compressive. Specifically, "uniaxial compression" means that one principal stress is compressive and one is tensile; "biaxial compression" means that both are compressive.

and external tensile load, will buckle, possibly causing failure of the plate. Houghton and Rothwell (16) have shown that in plates with holes having a very small gross curvature the stress concentration factor can increase appreciably. It is reasonable to assume then, that even though local distortions of a plate may be acceptable, such distortions could increase the stress concentration beyond that expected for a flat plate. Failure of the plate by yielding could then occur.

There is no known investigation of this buckling problem, other than an experimental investigation by Danis (14), to determine the critical stress for the buckling of a finite plate with a round hole.

This paper presents a theoretical investigation of the incipient buckling problem for an infinite plate containing a round hole. This problem is of interest in itself and for the practical reasons given in the preceding discussion. Results are presented on page 33.

2. PROBLEM DESCRIPTION

Consider a plate infinitely extended in two directions and pierced with a hole, as shown in Figure 1.

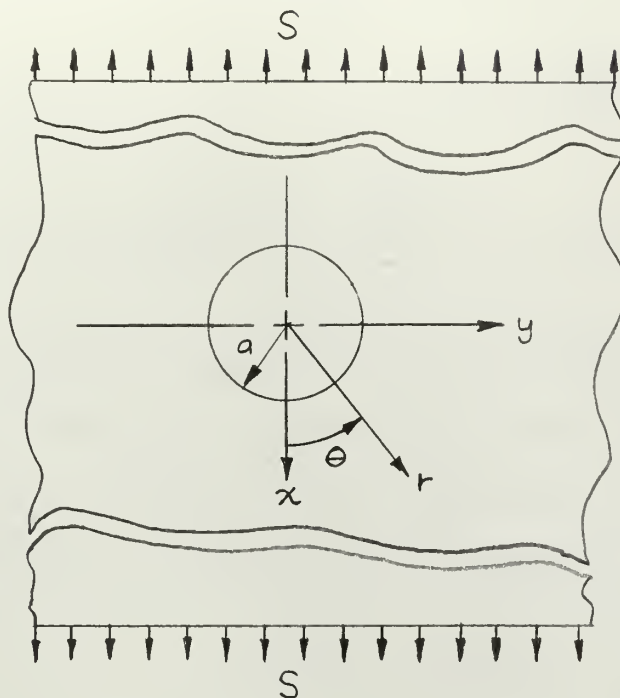


Figure 1

x and y are Cartesian coordinates, r and θ , cylindrical coordinates. The z axis, not shown, is perpendicular to the paper at the center of the hole, and extends in the positive direction toward the reader. Consider all quantities to be in units of hole radius, a , which essentially reduces them to dimensionless quantities, with hole radius equal to one.

The plate is placed in uniform tension, S psi, at infinity in the x direction. The solution of the stress distribution problem for this plate, as originally solved by Kirsch (19), and clearly presented by Timoshenko (7) is:

$$\sigma_r = \frac{S}{2} \left[\left(1 - \frac{1}{r^2} \right) + \left(1 + \frac{3}{r^4} - \frac{4}{r^2} \right) \cos 2\theta \right] \quad (1)$$

$$\sigma_\theta = \frac{S}{2} \left[\left(1 + \frac{1}{r^2} \right) - \left(1 + \frac{3}{r^4} \right) \cos 2\theta \right] \quad (2)$$

$$\tau_{r\theta} = \frac{S}{2} \left[1 - \frac{3}{r^4} + \frac{2}{r^2} \right] \sin 2\theta \quad (3)$$

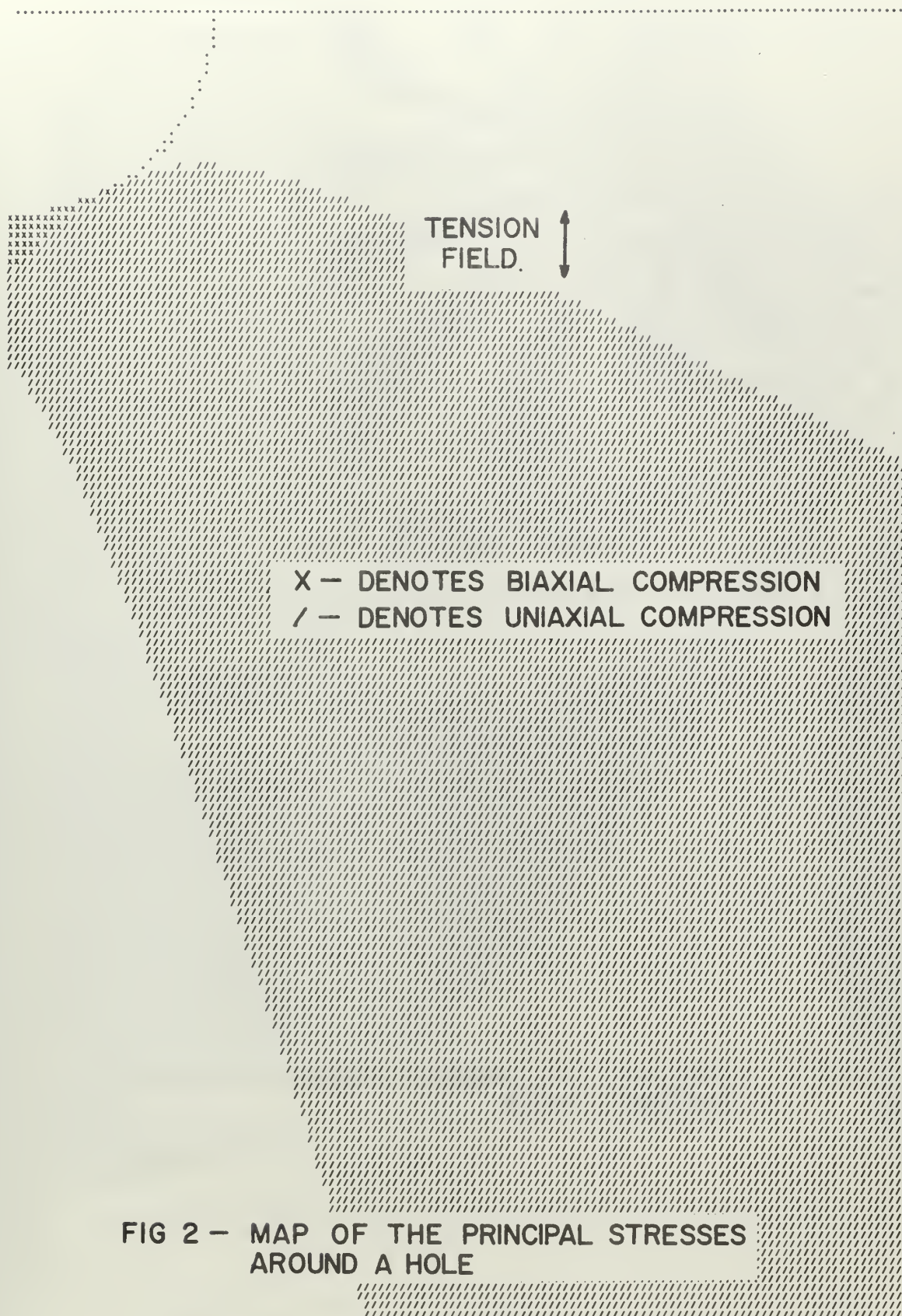
This set of equations is henceforth referred to as the Kirsch solution.

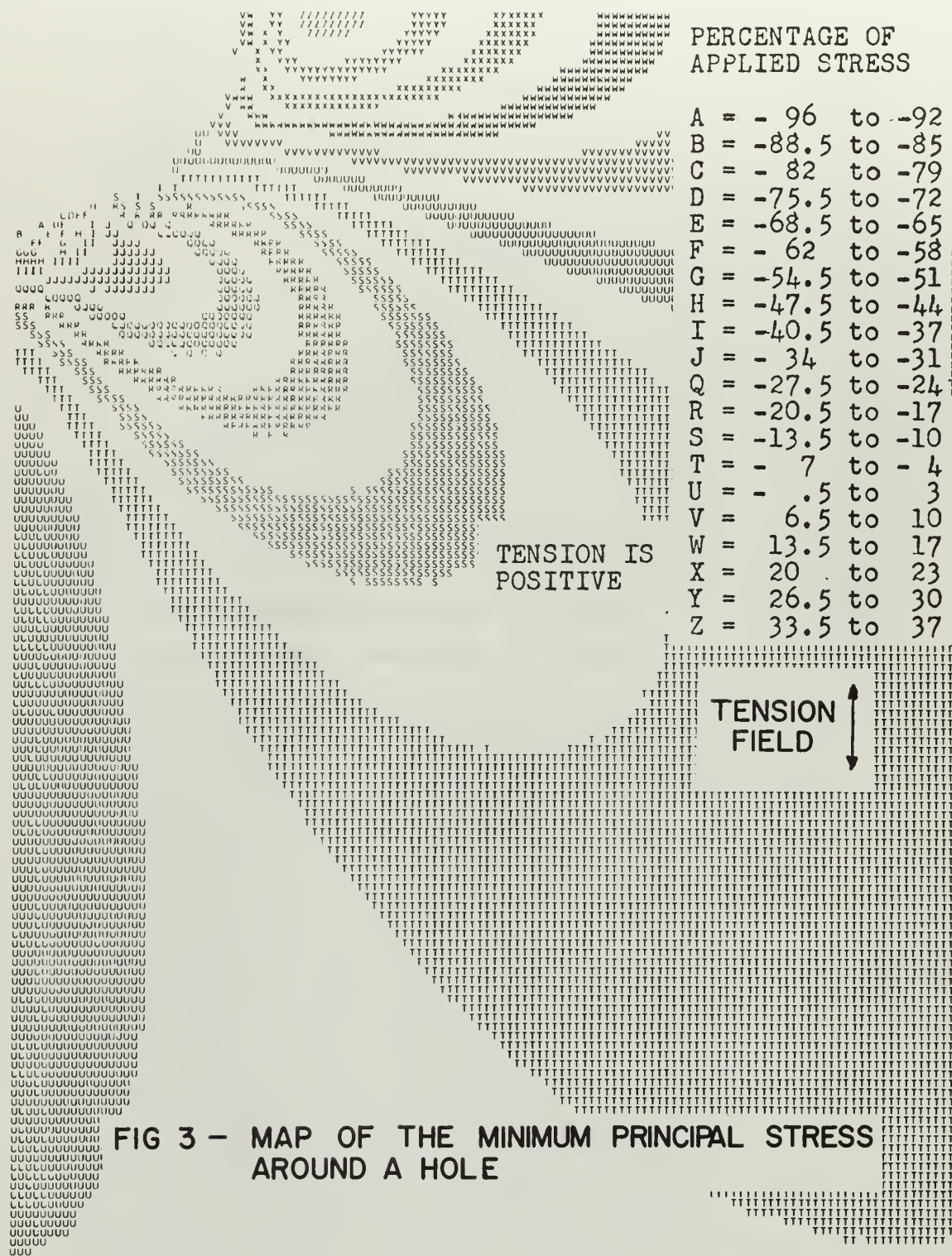
It is observed that the maximum tensile stress is 3 times the stress applied at infinity, and occurs at $r = 1$, and $\theta = \pm\pi/2$. The maximum compressive stress, equal in magnitude to the applied tensile stress, occurs at $r = 1$, and $\theta = \pm\pi$. From this it can be seen that if buckling were to occur in the plate, it would be most apparent near the hole along the x axis.

A better understanding of the buckling problem is obtained by considering the minimum principal stresses (most negative, largest compressive). From the Kirsch solution, the minimum principal stresses are found and shown in Figures 2 and 3.

Figure 2 shows one quarter of the plate, with the hole centered at the top left. The stress distribution is symmetrical with respect to the x and y axes. This picture was prepared with consideration only to whether the stress situation at a particular point is biaxial tensile, biaxial compressive, or uniaxial compressive.

Figure 3 shows the same section of plate as Figure 2. This presentation maps the relative values of the algebraic minimum principal stress at each point.





It can be seen from these figures that there is an area having biaxial compression. Accordingly, buckling is expected to take place for some sufficiently great value of applied load.

Numerous methods have been developed for solutions of plate buckling problems. In general, these methods involve either the satisfaction of the equilibrium equations or minimizing the strain energy of the plate (Energy Methods). A compilation of many solutions to plate buckling problems was made by Gerard and Becker (11). This is an excellent reference for both method and practical solutions.

One of the first energy methods was developed by Ritz (20), and independently, by Lord Rayleigh (4). This method is presented in a rigorous manner by Sokolnikoff (6). Budiansky and Hu (12) give a detailed example of the application of this method to a plate buckling problem.

The Rayleigh-Ritz method involves determining the total strain energy in the plate as a function of an assumed buckled configuration. Equilibrium, and the stable buckled deflection, $\bar{W}(x,y)$, exists when this strain energy, U_t , is minimized with respect to $W(x,y)$.

To do this, first, a family of functions, $\bar{\Phi}$, are assumed, which are dependent upon parameters A_k , such that the buckled deflection, $W(x,y)$, can be represented by:

$$W(x,y) = \bar{\Phi}(x,y,A_1,A_2,\dots) \quad (4)$$

where $\bar{\Phi}$ is such that the boundary conditions on the plate are satisfied for any values of the A_i .

The functions $\bar{\Phi}$ are substituted into the strain energy expression U_t so that

$$U_t = \iint_{\text{plate}} \left[\bar{\Phi}(x,y,A_1,A_2,\dots) \right]^2 dx dy \quad (5)$$

where U_t is the integrated total energy in the plate.

In order to minimize the energy function, the following set of equations is formed

$$\left. \begin{aligned} \frac{\partial U_t}{\partial A_1} &= 0 \\ \frac{\partial U_t}{\partial A_2} &= 0 \\ &\vdots \end{aligned} \right\} \quad (6)$$

If this set of equations has solutions $\bar{A}_1, \bar{A}_2, \dots$, the function

$$\bar{W}(x, y) = \bar{\Phi}(x, y, \bar{A}_1, \bar{A}_2, \dots)$$

represents the buckled equilibrium condition, the boundary conditions of the problem being satisfied by $\bar{\Phi}$ for any values of the A_i .

3. BOUNDARY CONDITIONS

For the problem of an infinite plate with a hole, two boundaries are of concern. At infinity, deflection, slope, and curvature are zero. Around the hole, a free edge condition exists.

The Kirsch solution is used, which already implies zero tractions around the hole, and uniaxial tension at infinity, so these conditions have already been satisfied.

For a free edge to exist around the hole means that there is no shear, twist, or moment applied at the edge of the hole. Following Timoshenko (8), and considering the cylindrical coordinate system established in Figure 1, the conditions for a free edge are

$$(Q_r)_{r=1} = 0 \quad (\text{shear} = 0) \quad (7)$$

$$(M_{rt})_{r=1} = 0 \quad (\text{twist} = 0) \quad (8)$$

$$(M_r)_{r=1} = 0 \quad (\text{moment} = 0) \quad (9)$$

Kirchoff (18) has shown that the twist condition can be reduced to a statically equivalent shear force. The original requirements of zero shear and zero twist can be reduced to one requirement such that the sum of the shear forces and the static equivalent of the twisting moments equals zero. The conditions then become

$$\left(Q_r - \frac{\partial M_{rt}}{\partial \theta} \cdot \frac{1}{r} \right)_{r=1} = 0 \quad (\text{Kirchoff shear} = 0) \quad (10)$$

$$(M_r)_{r=1} = 0 \quad (\text{moment} = 0) \quad (11)$$

where Q_r , M_r , and M_{rt} are expressed as

$$Q_r = -D \frac{\partial}{\partial r} \left[\frac{\partial^2 W}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial W}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2 W}{\partial \theta^2} \right] \quad (12)$$

$$M_r = -D \left[\frac{\partial^2 W}{\partial r^2} + \nu \left(\frac{1}{r} \cdot \frac{\partial W}{\partial r} + \frac{1}{r^2} \frac{\partial^2 W}{\partial \theta^2} \right) \right] \quad (13)$$

$$M_{rt} = D(1-\nu) \left[\frac{1}{r} \cdot \frac{\partial^2 W}{\partial r \partial \theta} - \frac{1}{r^2} \cdot \frac{\partial W}{\partial \theta} \right] \quad (14)$$

When the appropriate differentiations and substitutions are made,
the boundary condition equations become

$$\left[\frac{\partial^3 W}{\partial r^3} + \frac{1}{r} \cdot \frac{\partial^2 W}{\partial r^2} - \frac{1}{r^2} \cdot \frac{\partial W}{\partial r} + \frac{(2-\nu)}{r^2} \cdot \frac{\partial^3 W}{\partial r \partial \theta^2} - \frac{(3-\nu)}{r^3} \cdot \frac{\partial^2 W}{\partial \theta^2} \right]_{r=1} = 0 \quad \begin{matrix} \text{(Kirchoff} \\ \text{Shear} = 0) \end{matrix} \quad (15)$$

$$\left[\frac{\partial^2 W}{\partial r^2} + \frac{\nu}{r} \left(\frac{\partial W}{\partial r} + \frac{1}{r} \cdot \frac{\partial^2 W}{\partial \theta^2} \right) \right]_{r=1} = 0 \quad \text{(Moment} = 0) \quad (16)$$

$$\left[W = \frac{\partial W}{\partial r} = \frac{\partial^2 W}{\partial r^2} \right]_{r=\infty} = 0 \quad (17)$$

As previously noted, each member of the family of functions must satisfy the boundary conditions in the Rayleigh-Ritz method.

For a choice of Φ let

$$W(x,y) = \sum_{N=1}^{\infty} \sum_{M=0}^{\infty} r^{-N} \left[1 + \frac{X_{NM}}{r} + \frac{Y_{NM}}{r^2} \right] \left\{ \begin{matrix} A_{NM} \cos M\theta \\ \text{or} \\ B_{NM} \sin M\theta \end{matrix} \right\} \quad (18)$$

where A_{NM} and B_{NM} are arbitrary coefficients that are varied to minimize strain energy in the plate.

The boundary conditions of eq. 17 are satisfied by this series since at $r = \infty$, W and all of its derivatives are zero. Conditions of equations 15 and 16 are satisfied by substituting equation 18 into these two boundary condition equations (see Appendix A). Such a substitution results in two equations, 15 and 16, with two unknowns, X_{NM} AND Y_{NM} , for each combination of N and M . A unique value of X_{NM} and Y_{NM} is found which allows each term* of equation 18 to satisfy the free edge boundary conditions for any N and M chosen.

From Figures 1 and 3, it is seen that there is symmetry with respect to both the x and y axes. This means that when W is written as a Fourier series in θ (eq. 18), the series contains either cosine

* A term in this sense means the full expression

$$r^{-N} \left[1 + \frac{X_{NM}}{r} + \frac{Y_{NM}}{r^2} \right] \cdot \left[\begin{matrix} A_{NM} \cos M\theta \\ \text{or} \\ B_{NM} \sin M\theta \end{matrix} \right]$$

terms or sine terms, but not both. Additionally, the requirement of symmetry around both the x and y axes means that only odd or even terms in M need be taken. Thus the Fourier portion of equation 18 reduces to:

$$A_{NM} \cos M\theta, \text{ with } M = 1, 3, 5, \dots$$

or

$$A_{NM} \cos M\theta, \text{ with } M = 0, 2, 4, \dots$$

or

$$B_{NM} \sin M\theta, \text{ with } M = 1, 3, 5, \dots$$

or

$$B_{NM} \sin M\theta, \text{ with } M = 2, 4, 6, \dots$$

This gives four separate expressions for W, one for each possible condition of symmetry. The full Fourier series could be taken, but if this is done, three quarters of the coefficients, A_{NM} or B_{NM} are zero and considerable computer storage space is wasted. It is shown later that each of these series represents a different buckling mode.

4. ENERGY IN THE PLATE

To this point, a family of functions $W(r, \theta)$ (equation 18) have been developed, which satisfy the boundary conditions independently of the coefficients, A_{NM} or B_{NM} , and are adequate to represent possible shapes of the plate. Attention is now turned upon formulating an expression for the total strain energy, and then minimizing this expression to obtain the critical buckling stress.

The energy in the plate can be divided into two forms. First, there is the membrane energy which is the energy stored because the mid-plane is stretched. Second, there is the bending and twisting energy which results from the plate buckling out of its plane.

A basic assumption is that the plate deflection, $W(x, y)$, is infinitesimally small, and that the mid-plane strains may be represented by

$$\left. \begin{aligned} e_x &= \frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial W}{\partial x} \right)^2 \\ e_y &= \frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial W}{\partial y} \right)^2 \\ r_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial W}{\partial x} \cdot \frac{\partial W}{\partial y} \end{aligned} \right\} \quad (19)$$

Terms of higher order in W than these are neglected, since W is infinitesimal; those terms which are retained are essential for the calculation of bending energy.

In the above expressions $\frac{\partial u}{\partial x}$, etc., represent the stretching of the plate in its own plane. The terms $\frac{1}{2} \left(\frac{\partial W}{\partial x} \right)^2$, etc., represent the difference between a length of flat plate, and the corresponding straight line length when the plate is bent (difference between arc and chord).

Stresses in the x and y directions are composed of

(a) Membrane stresses of finite magnitude, which come from the Kirsch Solution and are denoted by σ_x , σ_y and τ_{xy} below; and (b) Infinitesimal bending and twisting stresses which do not appear explicitly below, but which are involved in the bending and twisting energy.

The normal forces per unit length (N_x, N_y) can be determined from

$N_x = \sigma_x h$, etc. The membrane energy in the plate is

$$U_{MEM} = \frac{1}{2} \iint (N_x e_x + N_y e_y + N_{xy} r_{xy}) dx dy \quad (20)$$

Substituting the strains (eqns. 19), and letting

$$U_{MEM} = U_1 + U_2 \quad (21)$$

where

$$U_1 = \frac{h}{2} \iint \left[\sigma_x \frac{\partial u}{\partial x} + \sigma_y \frac{\partial v}{\partial y} + \tau_{xy} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] dx dy \quad (22)$$

and

$$U_2 = \frac{h}{4} \iint \left[\sigma_x \left(\frac{\partial w}{\partial x} \right)^2 + \sigma_y \left(\frac{\partial w}{\partial y} \right)^2 + \tau_{xy} \cdot \frac{\partial w}{\partial x} \cdot \frac{\partial w}{\partial y} \right] dx dy \quad (23)$$

an expression for U_{MEM} is found in terms of the stresses, strains, and the deflection.

It can be seen that the membrane energy is composed of two terms. One, U_1 , relates the work done by the stresses, σ_x , etc., through a distance $\frac{\partial u}{\partial x}$, etc. The other term, U_2 , relates the work done by the stresses, σ_x , etc., through a distance $\frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2$, etc., as the plate bends out of plane.

At this point, the assumption is made that the Kirsch solution is still valid, even if the plate has buckled infinitesimally. This means that U_1 is independent of the lateral deflection, W , and can be treated as a constant. This assumption is justified by considering

that the problem is one of incipient buckling, not postbuckling. This assumption considerably simplifies the problem, because it uncouples the membrane stress distribution from the buckled deflection.

An expression for the bending and twisting energy, U_3 , is obtained by considering the moments in the plate and the respective rotations at a point as functions of plate deflection. Timoshenko (9) shows this expression to be;

$$U_3 = \frac{D}{2} \iint \left\{ \left[\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} \right]^2 - (2-\nu) \left[\frac{\partial^2 W}{\partial x^2} \cdot \frac{\partial^2 W}{\partial y^2} - \left| \frac{\partial^2 W}{\partial x \partial y} \right|^2 \right] \right\} dx dy \quad (24)$$

Adopting the notation $W_r = \frac{\partial W}{\partial r}$, $W_\theta = \frac{\partial W}{\partial \theta}$, etc., and transforming the previous energy equations to cylindrical coordinates, with the proper limits of integration for the problem of a hole in an infinite plate, equations 22, 23, and 24 become, respectively

$$U_1 = \text{CONSTANT} \quad (25)$$

$$U_2 = \frac{h}{4} \int_0^{2\pi} \int_1^\infty \left(\sigma_r W_r^2 + \frac{\sigma_\theta W_\theta^2}{r^2} + \frac{2 \tau_{r\theta} W_r W_\theta}{r} \right) r dr d\theta \quad (26)$$

$$U_3 = \frac{D}{2} \int_0^{2\pi} \int_1^\infty \left\{ \left[W_{rr} + \frac{W_r}{r} + \frac{W_{\theta\theta}}{r^2} \right]^2 - (2-\nu) \left[W_{rr} \left(\frac{W_r}{r} + \frac{W_{\theta\theta}}{r^2} \right) - \left| \frac{W_\theta}{r^2} - \frac{W_{r\theta}}{r} \right|^2 \right] \right\} r dr d\theta \quad (27)$$

σ_r, σ_θ and $\tau_{r\theta}$ are obtained from the Kirsch solution (eqns. 1, 2, and 3).

W and its derivatives are obtained from the assumed series (eq 18).

5. SOLUTION OF THE PROBLEM

With the expression for total energy ($U_t = U_1 + U_2 + U_3$) formulated, the next step is to substitute the Kirsch solution, and the assumed series for deflection and its derivatives. Then this expression must be minimized with respect to each of the unknown coefficients, A_{NM} or B_{NM} . The substitutions and integrations are shown in Appendices B, C, and D. It is sufficient for our purpose here to say that what arises from the integration of the expressions for U_2 and U_3 , are two quadruple series, with indices I and K summed over the range of N, and J and L summed over the range of M. The quadruple series stems from the multiplication of two double sums. Equations 26 and 27 now become*

$$U_2 = \frac{Sh}{4} \sum_I \sum_J \sum_K \sum_L Q_{IJKL} \cdot A_{IJ} \cdot A_{KL} \quad (28)$$

$$U_3 = \frac{D}{2} \sum_I \sum_J \sum_K \sum_L P_{IJKL} \cdot A_{IJ} \cdot A_{KL} \quad (29)$$

where the flexural rigidity of the plate, D, equals $\frac{Eh^3}{12(1-\nu^2)}$.

Q_{IJKL} and P_{IJKL} are different for each value of I, J, K, or L.

S is the stress applied to the plate at infinity and is factored from the Kirsch solution after those expressions are substituted into the energy integrals. A_{IJ} and A_{KL} are the unknown coefficients in the series expression for deflection (eq. 18).

Following the Rayleigh-Ritz procedure, U_t is differentiated with respect to each of the unknown coefficients, so that the

* The unknown coefficients for the cosine series, A_{NM} , only will be used henceforth, although these could be the sine series coefficients, B_{NM} , equally as well.

following set of simultaneous equations result,

$$\left. \begin{aligned} 0 &= \frac{\partial U_t}{\partial A_{II}} = \left[\sum_K \sum_L \left(\frac{S_h}{4} Q_{IIKL} + \frac{D}{2} P_{IIKL} \right) A_{KL} \right] + \left[\frac{S_h}{4} Q_{IIII} + \frac{D}{2} P_{IIII} \right] A_{II} \\ 0 &= \frac{\partial U_t}{\partial A_{I2}} = \left[\sum_K \sum_L \left(\frac{S_h}{4} Q_{I2KL} + \frac{D}{2} P_{I2KL} \right) A_{KL} \right] + \left[\frac{S_h}{4} Q_{I2I2} + \frac{D}{2} P_{I2I2} \right] A_{I2} \\ &\vdots \end{aligned} \right\} \quad (30)$$

The last term in each equation arises from the fact that

$$\frac{\partial}{\partial A_{IJ}} (A_{IJ} \cdot A_{IJ}) = 2 A_{IJ}.$$

As a practical matter, the equations arrived at in this investigation are too complicated to permit considering the use of a doubly infinite set of terms in the double series. Accordingly, it is henceforth presumed that truncated series are used and that there is but a finite number of unknown A_{NM} 's.

Simplification in notation is made at this point if the quadruple subscripts are replaced by a double subscript arrangement.

Let β equal the number of terms taken in M, and α equal the number of terms taken in N. Define

$$I' = (I-1)\beta + J$$

$$J' = (K-1)\beta + L$$

Using these equations a unique pair of subscripts can be changed to a unique single subscript, so that $A_{IJ} = A_{I'}$, and $Q_{IJKL} = Q_{I'J'}$, etc.

For example, if 4 terms were taken in M, and we wished to convert Q_{7492} to some doubly subscripted $Q_{I'J'}$, then $I = 7$, $J = 4$, $K = 9$, and $L = 2$; and

$$I' = (7-1) 4 + 4,$$

$$= 28$$

$$J' = (9-1) 4 + 2$$

$$= 34$$

so that Q_{7492} becomes $Q_{(28)(34)}$. Similarly

$$A_{11} \rightarrow A_1$$

$$A_{12} \rightarrow A_2$$

$$A_{13} \rightarrow A_3$$

etc.

$$Q_{1111} \rightarrow Q_{11}$$

$$Q_{1211} \rightarrow Q_{21}$$

$$Q_{1112} \rightarrow Q_{12}$$

etc.

Also, let

$$\bar{J} = (\alpha - 1) \cdot \beta + \beta = \alpha \cdot \beta$$

where $\bar{J}$ is the largest value of I' and J' respectively.

Substituting

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

and

$$R_{I'J'} = Q_{I'J'} \quad \text{and} \quad T_{I'J'} = P_{I'J'} \quad \text{if} \quad I' \neq J'$$

$$R_{I'J'} = 2 \cdot Q_{I'J'} \quad \text{and} \quad T_{I'J'} = 2 \cdot P_{I'J'} \quad \text{if} \quad I' = J'$$

equation 30 simplifies to

$$\left. \begin{aligned} \frac{\partial U_t}{\partial A_1} = 0 &= \sum_{J'} \left[\left(\frac{6S(1-\nu^2)}{Eh^2} \right) R_{1J'} + T_{1J'} \right] \\ \frac{\partial U_t}{\partial A_2} = 0 &= \sum_{J'} \left[\left(\frac{6S(1-\nu^2)}{Eh^2} \right) R_{2J'} + T_{2J'} \right] \\ &\vdots \\ \frac{\partial U_t}{\partial A_{\bar{J}}} = 0 &= \sum_{J'} \left[\left(\frac{6S(1-\nu^2)}{Eh^2} \right) R_{\bar{J}J'} + T_{\bar{J}J'} \right] \end{aligned} \right\} \quad (31)$$

When the substitution

$$\lambda = \frac{6S(1-\nu^2)}{Eh^2} \quad (32)$$

is made, and the above set of equations expressed in matrix notation, the following matrix eigenvalue problem results

$$(\underline{R}\lambda + \underline{I})\underline{A} = 0 \quad (33)$$

where

$$\underline{\underline{R}} = \begin{bmatrix} R_{11} & R_{12} & \cdot & \cdot & \cdot & R_{1\bar{j}} \\ R_{21} & R_{22} & & & & R_{2\bar{j}} \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ R_{\bar{j}1} & R_{\bar{j}2} & \cdot & \cdot & \cdot & R_{\bar{j}\bar{j}} \end{bmatrix} \quad (34)$$

$$\underline{\underline{T}} = \begin{bmatrix} T_{11} & T_{12} & \cdot & \cdot & \cdot & T_{1\bar{j}} \\ T_{21} & T_{22} & \cdot & \cdot & \cdot & T_{2\bar{j}} \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ T_{\bar{j}1} & T_{\bar{j}2} & \cdot & \cdot & \cdot & T_{\bar{j}\bar{j}} \end{bmatrix} \quad (35)$$

and

$$\underline{\underline{A}} = \begin{bmatrix} A_1 \\ A_2 \\ \cdot \\ \cdot \\ \cdot \\ A_{\bar{j}} \end{bmatrix} \quad (36)$$

The matrix $\underline{\underline{T}}$ is positive definite, since it arose from the bending energy integral. The $\underline{\underline{R}}$ matrix arose from the membrane energy integral, which contains terms from the Kirsch solution. This matrix is not necessarily positive definite. Both matrices are symmetric. These properties are important to the method utilized in solving the matrix eigenvalue problem (see Appendix D).

The question of whether differentiating U_t and setting these derivatives equal to zero, produces a minimum or a maximum, has not been discussed. Either situation can occur in the problem as it has been developed. Some of the eigenvalues obtained in a solution of equation 33 represent maxima which are not relevant to the plate buckling problem. The criterion for rejecting such answers is the set of equations

$$\frac{\partial^2 U_t}{\partial A_i^2} > 0$$

$$\left. \begin{aligned} \frac{\partial^2 U_t}{\partial A_2^2} &> 0 \\ &\vdots \\ \frac{\partial^2 U_t}{\partial A_{\bar{J}}^2} &> 0 \end{aligned} \right\} \quad (37)$$

If the set of equations 37 is satisfied, then U_t is minimized.

When the set of equations 31 is again differentiated with respect to the appropriate coefficient, the set of equations 37 become

$$\left. \begin{aligned} \frac{\partial^2 U_t}{\partial A_1^2} &= \lambda R_{11} + T_{11} > 0 \\ \frac{\partial^2 U_t}{\partial A_2^2} &= \lambda R_{22} + T_{22} > 0 \\ &\vdots \\ \frac{\partial^2 U_t}{\partial A_{\bar{J}}^2} &= \lambda R_{\bar{J}\bar{J}} + T_{\bar{J}\bar{J}} > 0 \end{aligned} \right\} \quad (38)$$

where $R_{I'I'}$ and $T_{I'I'}$ are the diagonal elements of the matrices $\underline{R}$ and $\underline{T}$, respectively. All of these terms are positive numbers, since they arise from the "squares" in the energy integrals (i.e., no cross product terms).

By rearranging equation 38 so that it becomes

$$\lambda > - \frac{T_{J'J'}}{R_{J'J'}}, \quad 1 \leq J' \leq \bar{J} \quad (39)$$

It is seen that for some negative values of λ , and all positive values, the solution of equation 33 represents a minimization of energy.

We are interested in only positive values, so that for any answer of interest, a minimization is assured. It should be noted that the negative values of λ that are admissible stem from

buckling modes in the plate that correspond to an applied compressive loading. These modes arise from the reversal of sign in the Kirsch solution and the associated reversed stress distribution around the hole. Modes of this type should be expected from a theoretical standpoint.

When a nontrivial solution has been found for equation 33, giving an eigenvalue λ and a corresponding eigenvector $\underline{A}$, then the buckling stress may be found by use of equation 32 and the buckled mode shape from equation 18.

6. RESULTS AND CONCLUSIONS

The results of this thesis may be presented compactly in the form of a formula for incipient buckling, derived from equation 37, viz.

$$S_{cr} = KE \left(\frac{t}{a} \right)^2 \quad (40)$$

where S_{cr} is the value of S , the tension at infinity, which causes incipient buckling, t is plate thickness, a is the hole radius and E is Young's modulus of elasticity. Values of the buckling coefficient, K , are given in table 1.

Values of K were obtained from a digital computer solution (see Appendix F), for the particular case

$$E = 30 \times 10^6 \text{ psi} \quad (\text{Young's Modulus})$$

$$\nu = .3 \quad (\text{Poisson's ratio})$$

$$h = .001 \text{ hole radii}$$

Equation 40 permits prediction of S_{cr} for other values of t/a and E . However, the effect of different values of ν cannot be studied easily since ν appears in the bending energy integral and in the boundary conditions. All results given are for $\nu = .3$, but the computer program can be modified by changing card listed as 0011 on page 84.

Four distinct mode shapes are apparent, as determined by different buckling coefficients appearing for each. A tabulation of K is shown in Table 1 below.

Table 1

Values of the Buckling Coefficient, K

<u>MODE</u>	<u>K</u>
1	7.65
2	7.80
3	11.7
4	12.3

Since the energy method involves the use of a series which cannot be made infinite in the present method, the actual buckling coefficients obtained will always be higher than the true value. This is because the deflection of the plate is only approximated, and the energy never quite minimized. A solution with less plate energy could be found, if more terms were taken in the series for deflection.

Different values of the buckling coefficient, K, actually calculated by use of the digital computer program are shown in Table 2, and are plotted against the number of terms taken in both the N and M parameters of the deflection series, equation 18. These plots are shown in Figures 4a through 4e. The projected values listed in Table 1 are taken graphically from these figures as the asymptotes which the curves for buckling coefficient appear to approach, when the number of terms in N and M increase. These projected values are considered to be the best evaluation of the buckling coefficient.

Figure 4f is taken from the experimental data of Danis (14). All of his points for a finite plate are plotted as K versus d/b,

Table 2

Values of the Buckling Coefficient, K, for Various
Numbers of Terms in the Series

Mode 1		Terms in M					
		4	5	6	7	8	9
Terms in N	6				11.258		
	8	9.9897	8.6493	8.4420	8.4173	8.4157	8.4153
	10	9.1333	8.0673	7.8967	7.8763	7.8748	7.8746
	12	8.9990	7.9743	7.8060	7.7857	7.7840	
	14	8.9927	7.9683	7.7987	7.7777		
	16	8.9750	7.9537	7.7847			
	18	8.9347	7.9237				
	20	8.8897	7.8913				
Mode 2		Terms in M					
		4	5	6	7	8	9
Terms in N	8	9.1250	8.5877	8.5107	8.5033	8.5026	8.5026
	10	8.5133	8.0630	7.9977	7.9913	7.9906	7.9906
	12	8.4310	7.9860	7.9197	7.9130	7.9127	
	14	8.4253	7.9773	7.9100	7.9030		
	16	8.4023	7.9577	7.8907			
	18	8.3637	7.9257				
	20	8.3253	7.8940				
Mode 3		Terms in M					
		4	5	6	7	8	9
Terms in N	8		15.592	14.545	14.371	14.348	14.347
	10		13.327	12.616	12.501	12.487	12.486
	12	16.288	12.679	12.046		11.929	
	14	15.959	12.516	11.897	11.793		
	16	15.910	12.493	11.872			
	18	15.907	12.488				
	20	15.882	12.470				
Mode 4		Terms in M					
		4	5	6	7	8	9
Terms in N	8		16.418	15.765	15.660	15.647	15.646
	10		13.714	13.318	13.258	13.252	13.252
	12	14.761	12.929	12.589		12.533	
	14	14.463	12.719	12.388	12.337		
	16	14.415	12.682	12.350			
	18	14.412	12.677				
	20	14.392	12.662				

the ratio of hole diameter to plate width. The buckling coefficients in Table 1, for modes 1 and 2, are plotted on the y axis (for zero d/b).

The values for modes 1 and 2 are in fair agreement with experimental data. No correlation between the theoretical prediction and the experimental observation is attempted for modes 3 and 4.

The buckled deflection of the plate for modes 1 through 4 is shown in Figures 5 through 8 respectively. It is seen in these figures that the four possible conditions of symmetry are satisfied by the four modes. This result is given in Table 3.

Table 3

Resultant Symmetry for the Various Mode Shapes

MODE	SYMMETRY WITH RESPECT TO X AXIS	SYMMETRY WITH RESPECT TO Y AXIS
1	$f(y) = f(-y)$	$f(x) = f(-x)$
2	$f(y) = f(-y)$	$f(x) = -f(-x)$
3	$f(y) = -f(-y)$	$f(x) = f(-x)$
4	$f(y) = -f(-y)$	$f(x) = -f(-x)$

In looking at Figure 4^c, it is apparent that mode 2 is converging to a value of buckling coefficient that is somewhat higher than that for mode 1. This can be qualitatively justified by considering the symmetry of buckled shape, and the equivalent one dimensional problem.

Consider two beams, buckled by some unspecified loading, into shapes as shown in Figure 9.

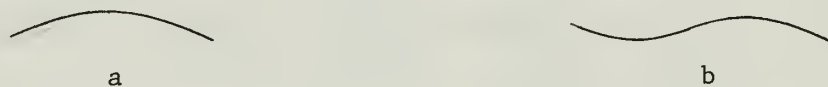


Figure 9

It is known from beam theory that the shape of Figure 9b has a higher critical buckling load than the shape of Figure 9a. This is exactly the relationship between modes 1 and 2 of the buckled plate, as can be seen in Figures 5 and 6, respectively.

There is a small area near the intersection of the hole edge and the y axis, which assumes cross sectional shapes similar to that in Figure 9a for mode 1, and in Figure 9b for mode 2. It is only in these small areas connecting the "top" of the plate to the "bottom", that this sort of bending occurs, so that the disparity between the buckling coefficients of the first and second plate modes is much smaller than that for the beam modes.

This same argument applies to the difference between modes 3 and 4, and to the gross differences between modes 3 and 4, and modes 1 and 2.

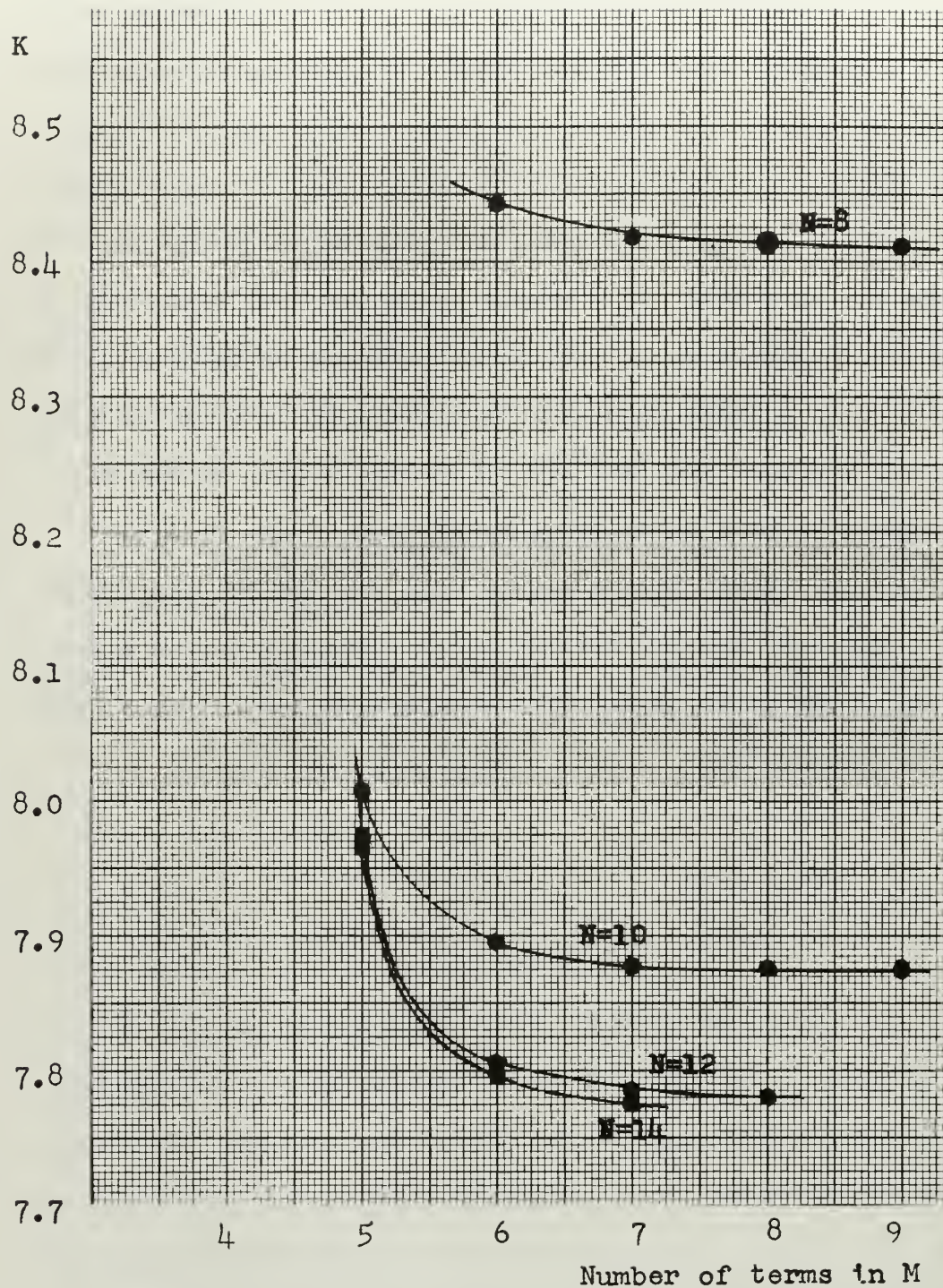


FIGURE 4a - Plot of the Buckling Coefficient K vs the number of terms taken in M for mode 1.

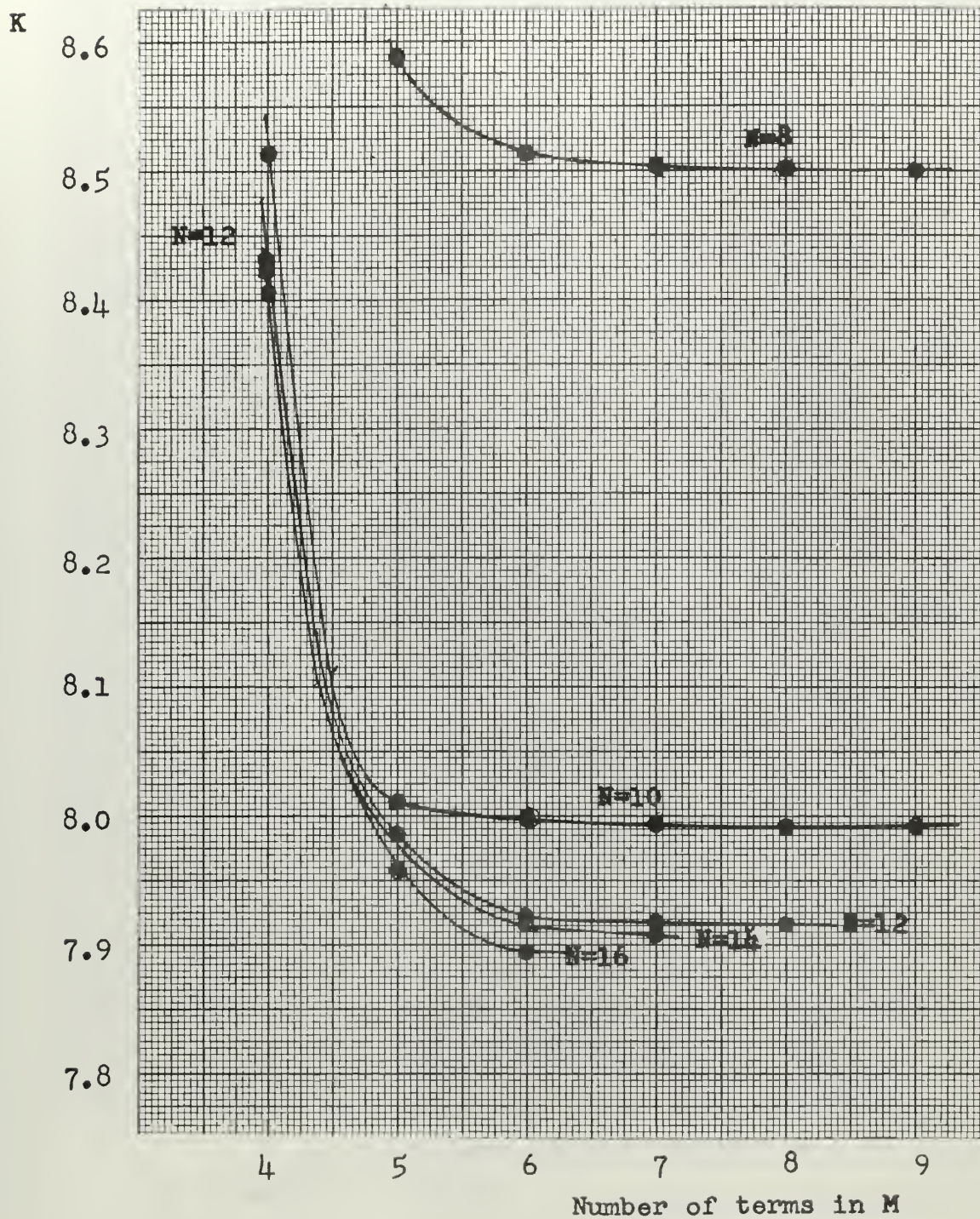


FIGURE 4b - Plot of the Buckling Coefficient K vs the number of terms taken in M , for mode 2.

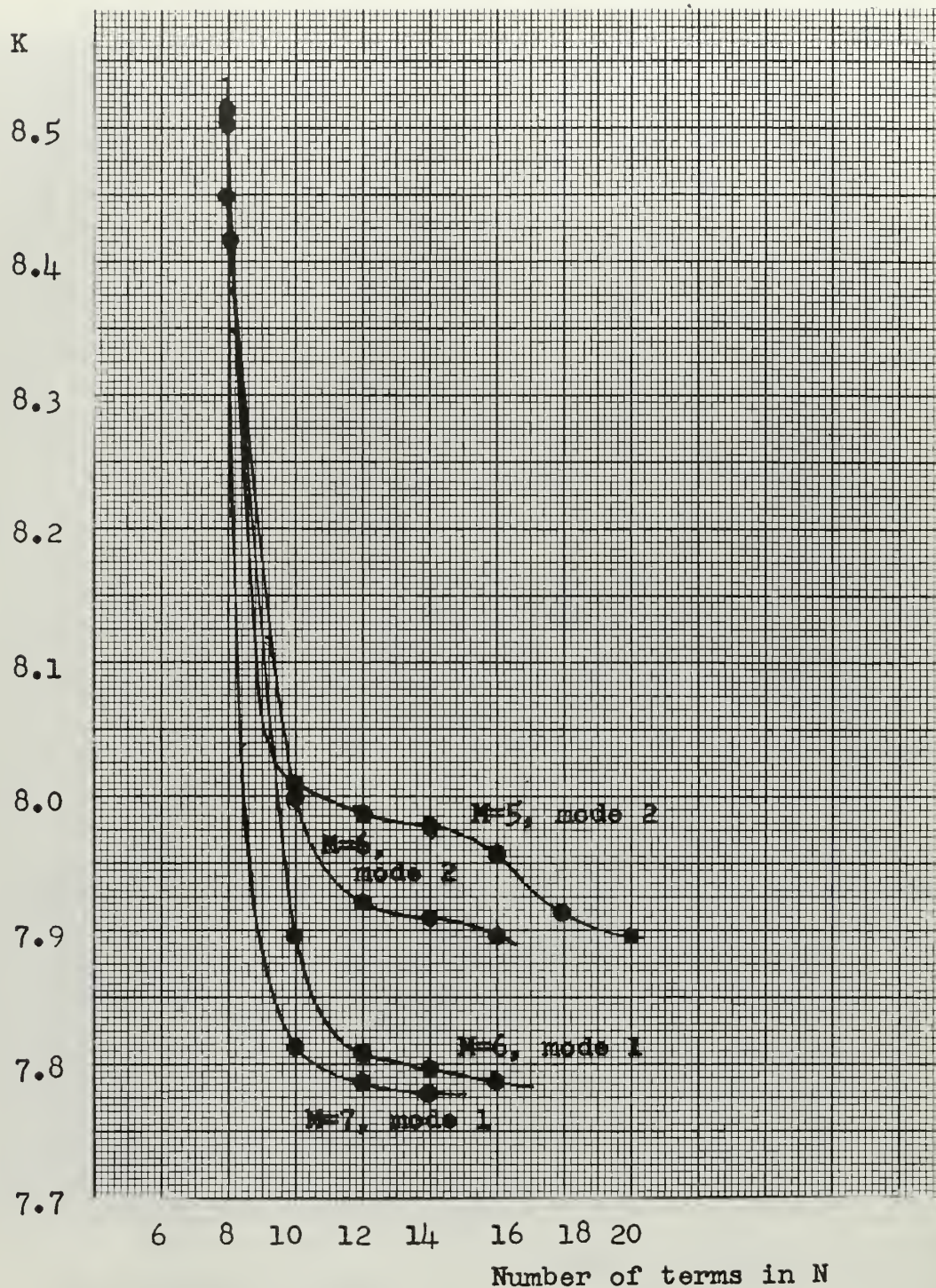


FIGURE 4c - Plot of the Buckling Coefficient K vs the number of terms taken in N , for modes 1 and 2

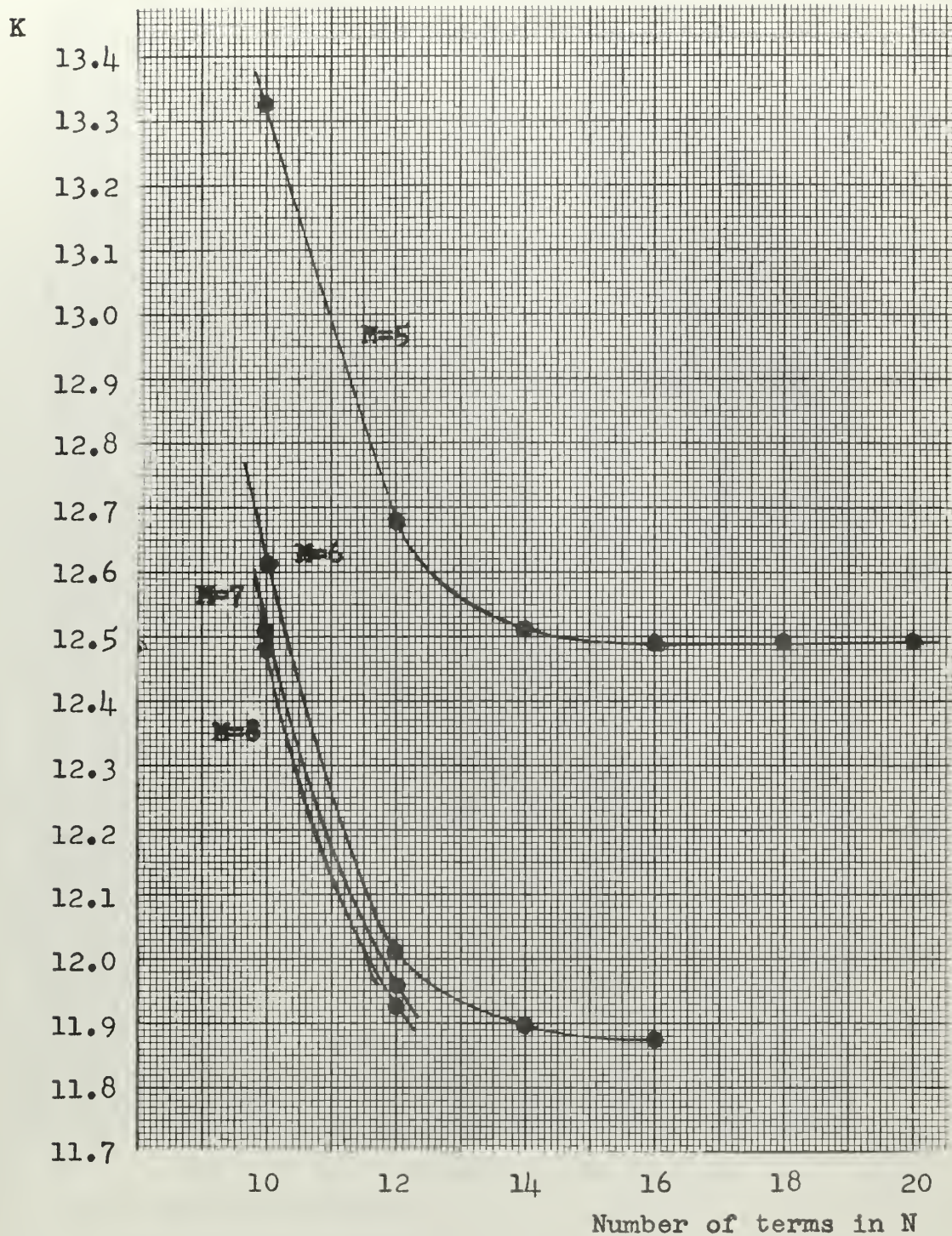


FIGURE 4d - Plot of the Buckling Coefficient K vs the number of terms taken in N for mode 3.

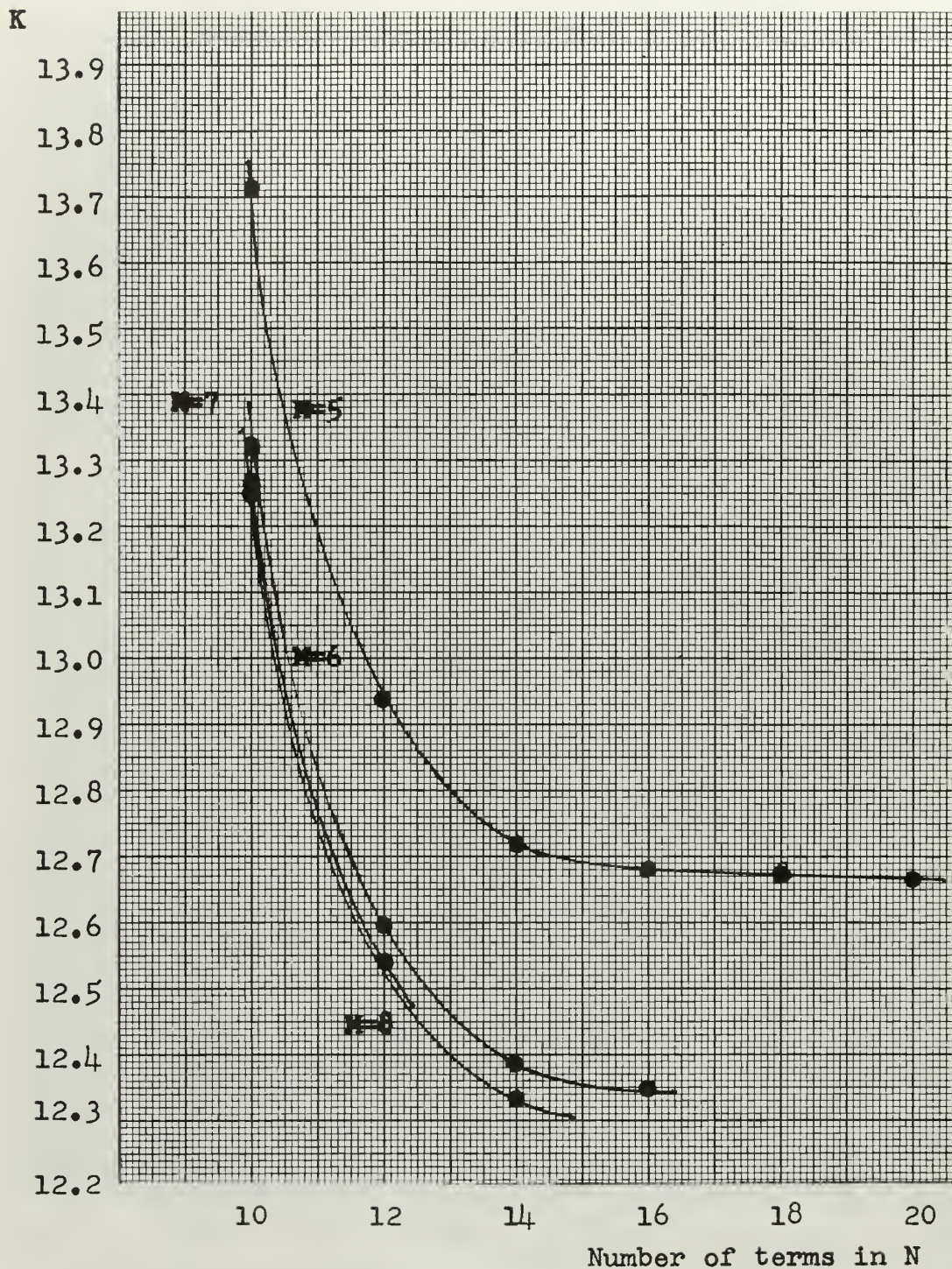


FIGURE 4_e - Plot of the Buckling Coefficient K vs the number of terms taken in N , for mode 4.

K

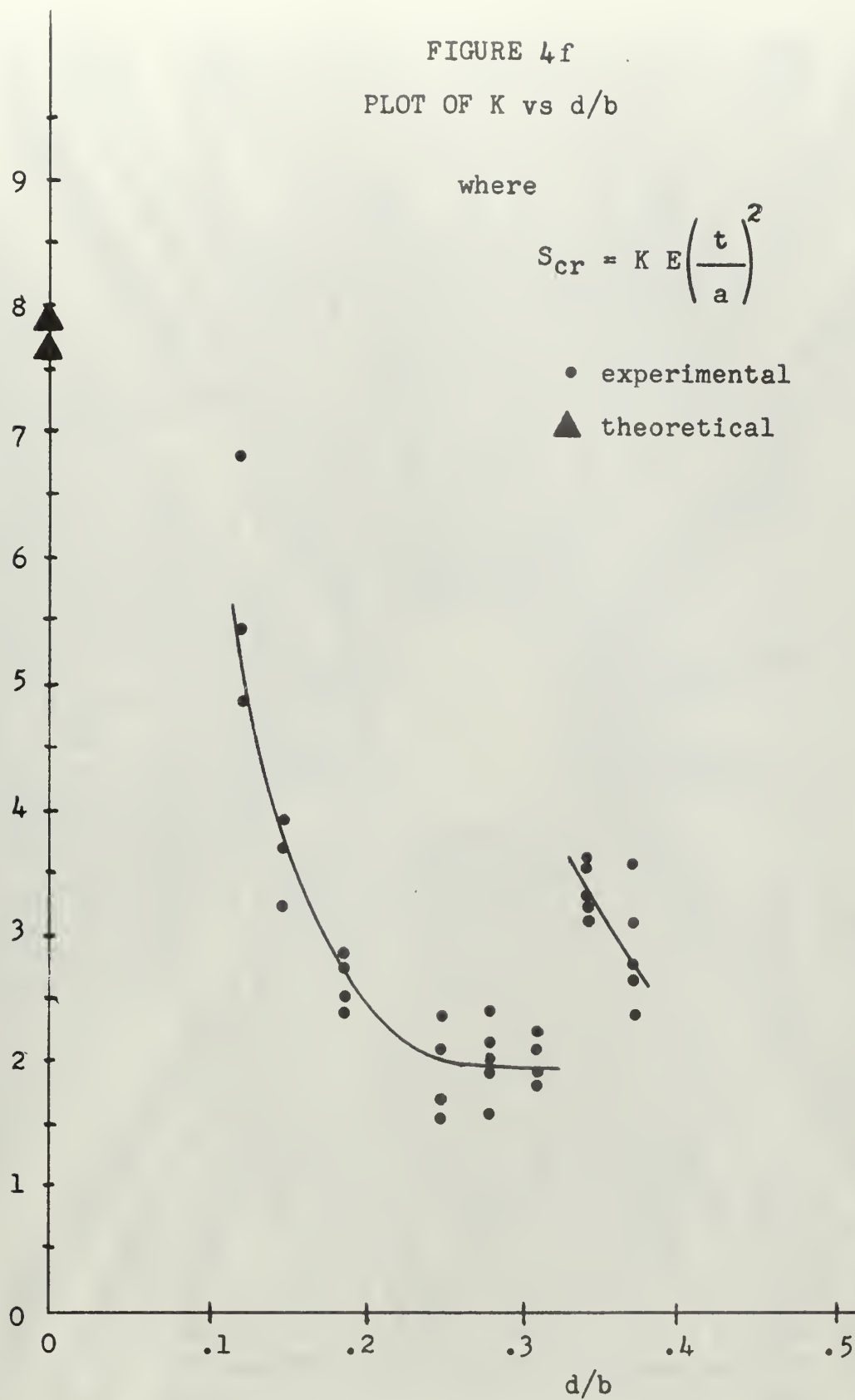
FIGURE 4f
PLOT OF K vs d/b

where

$$S_{cr} = K E \left(\frac{t}{a} \right)^2$$

• experimental

▲ theoretical



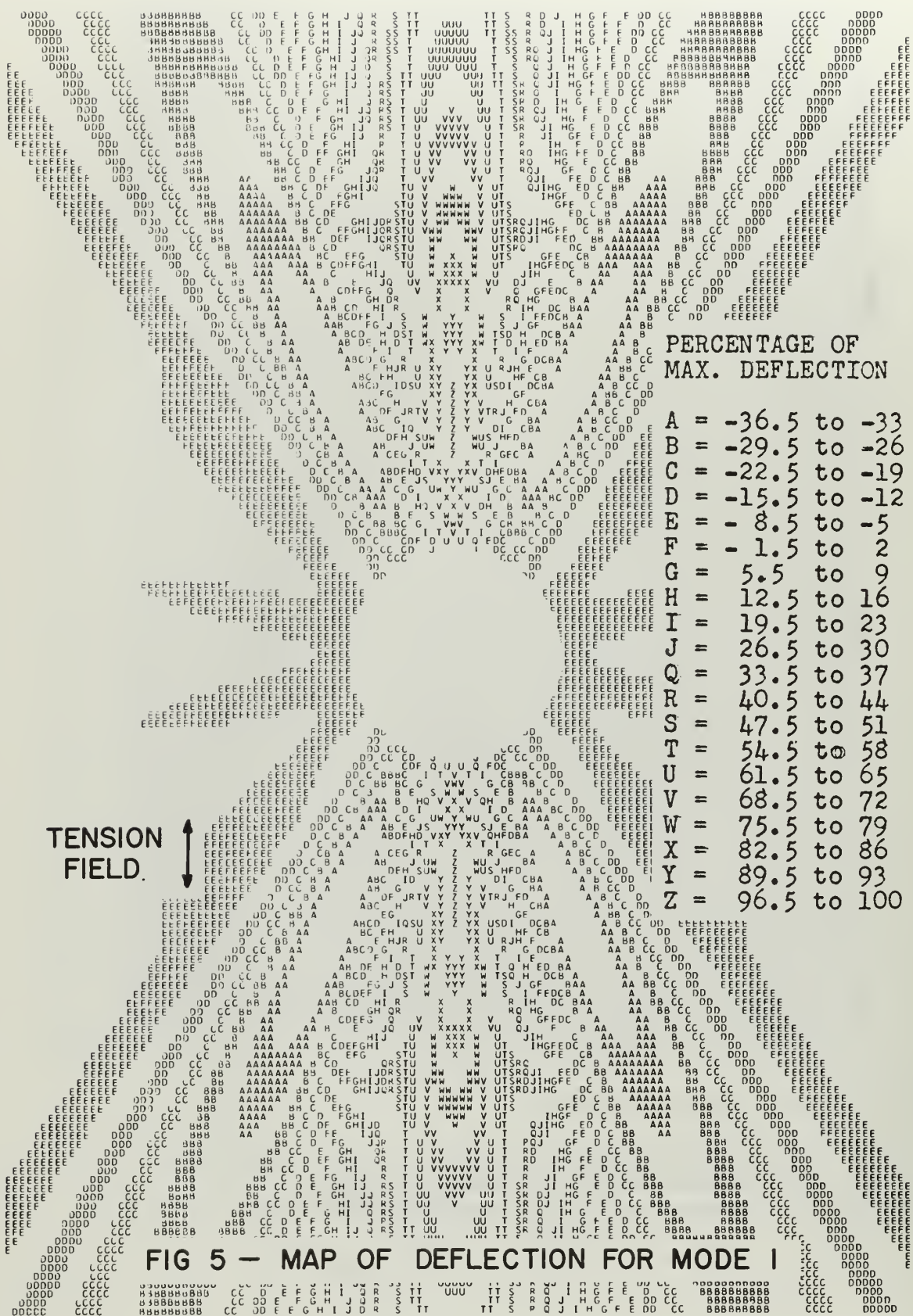
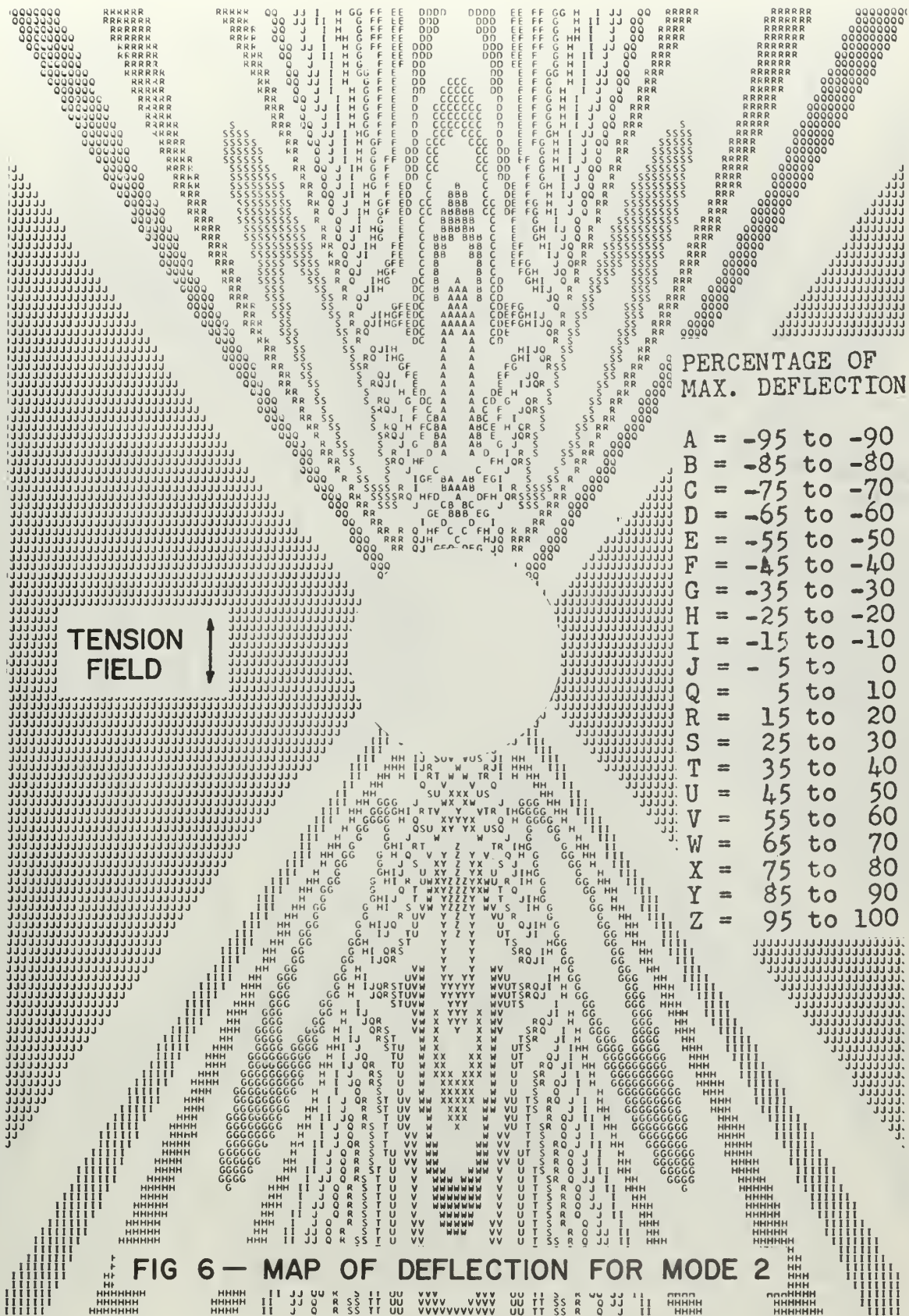
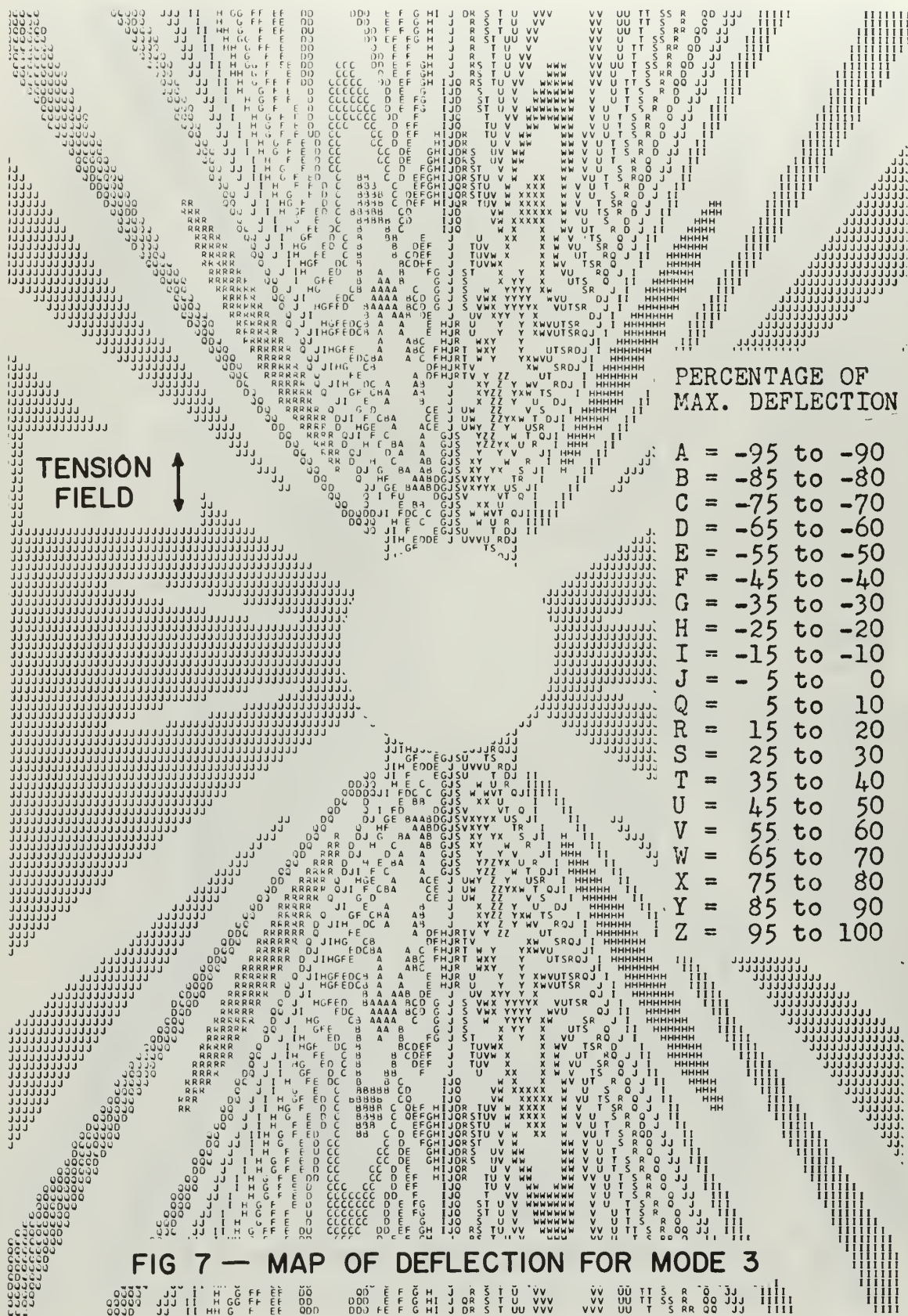
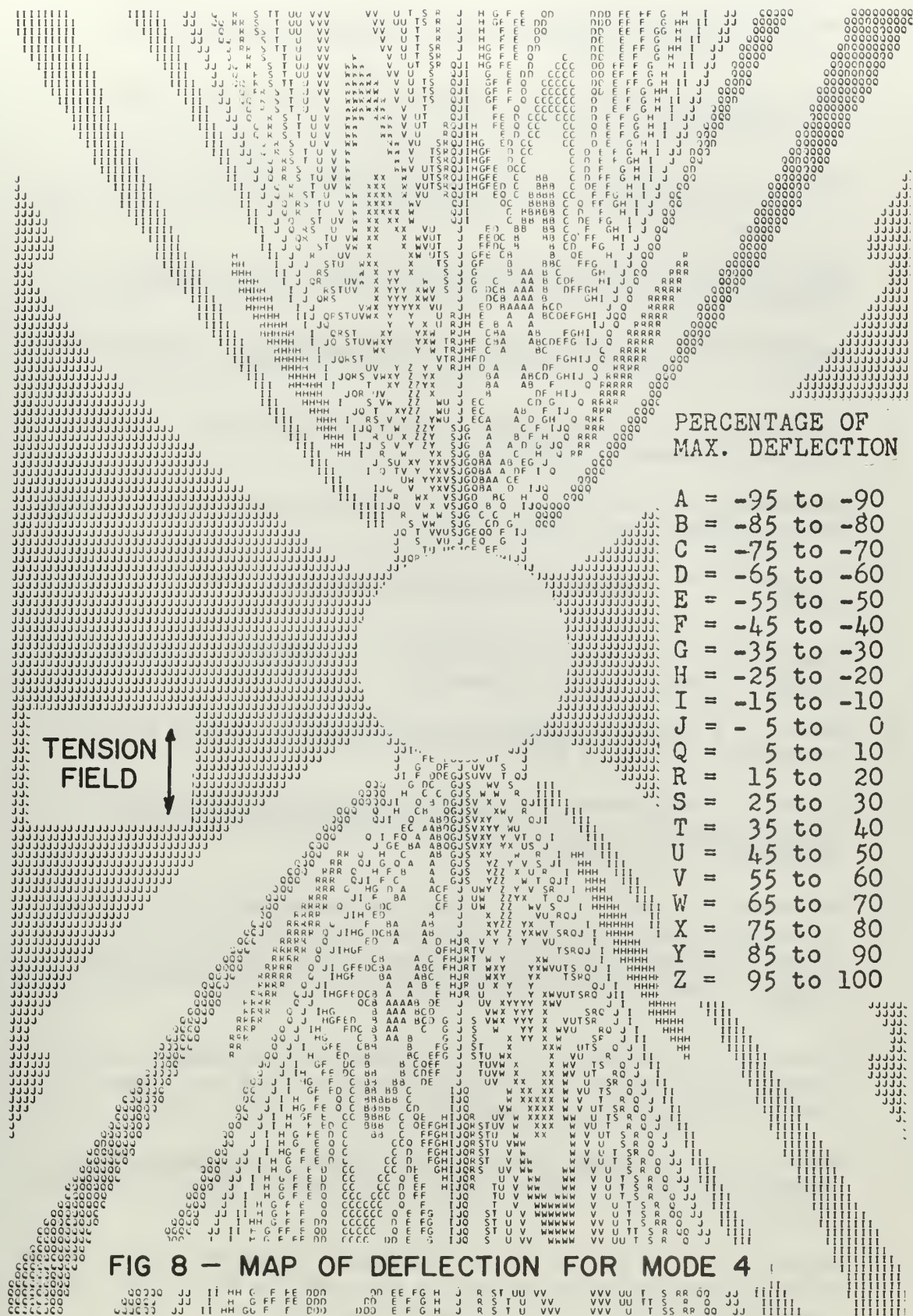


FIG 5 — MAP OF DEFLECTION FOR MODE I







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APPENDIX A

SATISFACTION OF THE BOUNDARY CONDITIONS

The free edge boundary conditions for the hole are stated in equations 15 and 16 as

$$\left[W_{rrr} + \frac{W_{rr}}{r} - \frac{W_r}{r^2} + \frac{(2-\nu)}{r^2} \cdot W_{r\theta\theta} - \frac{(3-\nu)}{r^3} \cdot W_{\theta\theta} \right]_{r=1} = 0 \quad (A-1)$$

and

$$\left[W_{rr} + \frac{\nu}{r} \left(W_r + \frac{W_{\theta\theta}}{r} \right) \right]_{r=1} = 0 \quad (A-2)$$

The plate deflection, W , is taken as

$$W(r, \theta) = \sum_{N=1}^{\infty} \sum_{M=0}^{\infty} r^{-N} \left[1 + \frac{X_{NM}}{r} + \frac{Y_{NM}}{r^2} \right] \begin{bmatrix} A_{NM} \cos M\theta \\ \text{or} \\ B_{NM} \sin M\theta \end{bmatrix} \quad (A-3)$$

where the coefficients A_{NM} and B_{NM} are varied to minimize the plate energy, and the coefficients X_{NM} and Y_{NM} are chosen so that the boundary conditions expressed by equations A-1 and A-2 are satisfied.

The derivatives of W are tabulated in Table 4. When these derivatives are substituted into equations A-1 and A-2, and evaluated at $r = 1$, the following equations result

$$0 = \sum_{N=1}^{\infty} \sum_{M=0}^{\infty} (S_1 + S_2 X_{NM} + S_3 Y_{NM}) \begin{bmatrix} A_{NM} \cos M\theta \\ \text{or} \\ B_{NM} \sin M\theta \end{bmatrix} \quad (A-4)$$

and

$$0 = \sum_{N=1}^{\infty} \sum_{M=0}^{\infty} (t_1 + t_2 X_{NM} + t_3 Y_{NM}) \begin{bmatrix} A_{NM} \cos M\theta \\ \text{or} \\ B_{NM} \sin M\theta \end{bmatrix} \quad (A-5)$$

where

$$S_1 = N(N+1) - N\gamma - M^2\gamma \quad (A-6)$$

$$S_2 = (N+1)(N+2) - (N+1)\gamma - M^2\gamma \quad (A-7)$$

$$S_3 = (N+2)(N+3) - (N+2)\gamma - M^2\gamma \quad (A-8)$$

$$t_1 = -N(N+1)(N+2) + N(N+1) + N + (2-\gamma)M^2N + (3-\gamma)M^2 \quad (A-9)$$

$$t_2 = -(N+1)(N+2)(N+3) + (N+1)(N+2) + (N+1) + (2-\gamma)M^2(N+1) + (3-\gamma)M^2 \quad (A-10)$$

$$t_3 = -(N+2)(N+3)(N+4) + (N+2)(N+3) + (N+2) + (2-\gamma)M^2(N+2) + (3-\gamma)M^2 \quad (A-11)$$

Equations A-4 and A-5 must hold for all values of θ , which can only be true if

$$S_1 + S_2 X_{NM} + S_3 Y_{NM} = 0 \quad (A-12)$$

$$t_1 + t_2 X_{NM} + t_3 Y_{NM} = 0 \quad (A-13)$$

for any value of N and M . Thus, a value of X_{NM} and Y_{NM} can be found for each value of N and M , such that the boundary conditions are satisfied.

By application of Cramer's Rule:

$$X_{NM} = \frac{\begin{vmatrix} -S_1 & S_3 \\ -t_1 & t_3 \end{vmatrix}}{\begin{vmatrix} S_2 & S_3 \\ t_2 & t_3 \end{vmatrix}} = \frac{-S_1 t_3 + S_3 t_1}{S_2 t_3 - S_3 t_2}$$

and

$$Y_{NM} = \frac{\begin{vmatrix} S_2 & -S_1 \\ t_2 & -t_1 \end{vmatrix}}{\begin{vmatrix} S_2 & S_3 \\ t_2 & t_3 \end{vmatrix}} = \frac{-S_2 t_1 + S_1 t_2}{S_2 t_3 - S_3 t_2}$$

The above results are incorporated into SUBROUTINE BOUND (see APPENDIX F) which generates values of X_{NM} and Y_{NM} for given values of N and M.

Table 4

Tabulation of the Derivatives of W

$$\begin{aligned}
 W &= \sum_{N=1}^{\infty} \sum_{M=0}^{\infty} \left[r^{-N} + X_{NM} r^{-(N+1)} + Y_{NM} r^{-(N+2)} \right] \left\{ \begin{matrix} A_{NM} \cos M\theta \\ \text{or} \\ B_{NM} \sin M\theta \end{matrix} \right\} \\
 W_r &= \sum_{N=1}^{\infty} \sum_{M=0}^{\infty} \left[-N r^{-(N+1)} - (N+1) X_{NM} r^{-(N+2)} - (N+2) Y_{NM} r^{-(N+3)} \right] \left\{ \begin{matrix} A_{NM} \cos M\theta \\ \text{or} \\ B_{NM} \sin M\theta \end{matrix} \right\} \\
 W_{rr} &= \sum_{N=1}^{\infty} \sum_{M=0}^{\infty} \left[N(N+1) r^{-(N+2)} + (N+1)(N+2) X_{NM} r^{-(N+3)} + (N+2)(N+3) Y_{NM} r^{-(N+4)} \right] \left\{ \begin{matrix} A_{NM} \cos M\theta \\ \text{or} \\ B_{NM} \sin M\theta \end{matrix} \right\} \\
 W_{rrr} &= \sum_{N=1}^{\infty} \sum_{M=0}^{\infty} \left[-N(N+1)(N+2) r^{-(N+3)} - (N+1)(N+2)(N+3) X_{NM} r^{-(N+4)} - (N+2)(N+3)(N+4) Y_{NM} r^{-(N+5)} \right] \left\{ \begin{matrix} A_{NM} \cos M\theta \\ \text{or} \\ B_{NM} \sin M\theta \end{matrix} \right\} \\
 W_{r\theta} &= \sum_{N=1}^{\infty} \sum_{M=0}^{\infty} \left[-N r^{-(N+1)} - (N+1) X_{NM} r^{-(N+2)} - (N+2) Y_{NM} r^{-(N+3)} \right] M \left\{ \begin{matrix} -A_{NM} \sin M\theta \\ \text{or} \\ B_{NM} \cos M\theta \end{matrix} \right\} \\
 W_{r\theta\theta} &= \sum_{N=1}^{\infty} \sum_{M=0}^{\infty} \left[N r^{-(N+1)} + (N+1) X_{NM} r^{-(N+2)} + (N+2) Y_{NM} r^{-(N+3)} \right] M^2 \left\{ \begin{matrix} A_{NM} \cos M\theta \\ \text{or} \\ B_{NM} \sin M\theta \end{matrix} \right\} \\
 W_{\theta\theta} &= \sum_{N=1}^{\infty} \sum_{M=0}^{\infty} \left[-r^{-2} - X_{NM} r^{-(N+1)} - Y_{NM} r^{-(N+2)} \right] M^2 \left\{ \begin{matrix} A_{NM} \cos M\theta \\ \text{or} \\ B_{NM} \sin M\theta \end{matrix} \right\} \\
 W_{\theta} &= \sum_{N=1}^{\infty} \sum_{M=0}^{\infty} \left[r^{-2} + X_{NM} r^{-(N+1)} + Y_{NM} r^{-(N+2)} \right] M \left\{ \begin{matrix} -A_{NM} \sin M\theta \\ \text{or} \\ B_{NM} \cos M\theta \end{matrix} \right\}
 \end{aligned}$$

APPENDIX B

EVALUATION OF THE ENERGY INTEGRALS

The total energy in the plate, from equations 25, 26, and 27, is:

$$U_t = U_1 + U_2 + U_3$$

where

$$U_1 = \text{constant}$$

$$U_2 = \frac{h}{4} \int_0^{2\pi} \int_1^\infty \left[\sigma_r W_r^2 + \frac{\sigma_\theta W_\theta^2}{r^2} + \frac{2\tau_{r\theta} W_r W_\theta}{r} \right] r dr d\theta \quad (\text{B-1})$$

$$U_3 = \frac{D}{2} \int_0^{2\pi} \int_1^\infty \left\{ \left[W_{rr} + \frac{W_r}{r} + \frac{W_{\theta\theta}}{r^2} \right]^2 - 2(1-\nu) \left[W_{rr} \left(\frac{W_r}{r} + \frac{W_{\theta\theta}}{r^2} \right) - \left(\frac{W_\theta}{r^2} - \frac{W_{r\theta}}{r} \right)^2 \right] \right\} r dr d\theta \quad (\text{B-2})$$

Note that these integrals contain second degree terms in the derivatives of the deflection, W. Since W is represented by a series, care must be taken to insure that the series are properly multiplied.

The product of two double sums, both summed over the same range for the subscripts N and M, is a quadruple series, with dummy subscripts I and K summed over the range of N, and K and L, summed over the range of M.

For example;

$$\begin{aligned} \left(\sum_{N=1}^{\infty} \sum_{M=0}^{\infty} (1+r^{-N}) \cos M\theta \right)^2 &= \left(\sum_{I=1}^{\infty} \sum_{J=0}^{\infty} (1+r^{-I}) \cos J\theta \right) \left(\sum_{K=1}^{\infty} \sum_{L=0}^{\infty} (1+r^{-K}) \cos L\theta \right) \\ &= \sum_{I=1}^{\infty} \sum_{J=0}^{\infty} \sum_{K=1}^{\infty} \sum_{L=0}^{\infty} (1+r^{-I})(1+r^{-K}) \cos J\theta \cos L\theta \\ &= \sum \sum \sum \sum (1+r^{-I}+r^{-K}+r^{-(I+K)}) \cos J\theta \cos L\theta \end{aligned}$$

It is seen that the cross product terms above do not combine directly into one term. One cross product term involves r^{-I} ; the other involves r^{-K} . For this reason, cross product terms in the energy integrals B-1 and B-2 must be identified and separated.

As an aid in keeping the cross product terms separated, the notation is adopted that starred (*) functions are functions of I and J, and that non-starred functions are functions of K and L, in place of N and M, respectively. Equation B-1 may be written

$$U_2 = \frac{h}{2} \int_0^{2\pi} \int_1^\infty \left[\sigma_r W_r^* W_r + \frac{\sigma_\theta W_\theta W_\theta^*}{r^2} + \frac{1}{r} T_{r\theta} (W_r^* W_\theta + W_r W_\theta^*) \right] r dr d\theta \quad (B-3)$$

and with the substitution of the Kirsch solution becomes

$$U_2 = \frac{Sh}{4} \int_0^{2\pi} \int_1^\infty \left[\left(1 - \frac{1}{r^2} \right) W_r^* W_r + \left(1 + \frac{3}{r^4} - \frac{4}{r^2} \right) W_r^* W_r \cos 2\theta + \left(1 + \frac{1}{r^2} \right) \frac{W_\theta^* W_\theta}{r^2} - \left(1 + \frac{3}{r^4} \right) \frac{W_\theta^* W_\theta}{r^2} \cos 2\theta - \left(1 - \frac{3}{r^4} + \frac{2}{r^2} \right) \cdot (W_\theta^* W_r + W_\theta W_r^*) \frac{\sin 2\theta}{r} \right] r dr d\theta \quad (B-4)$$

Equation B-2 is now

$$U_3 = \frac{D}{2} \int_0^{2\pi} \int_1^\infty (U_{3a} + U_{3b} + U_{3c}) r dr d\theta \quad (B-5)$$

where

$$U_{3a} = \left(W_{rr} + \frac{W_r}{r} + \frac{W_{\theta\theta}}{r^2} \right) \left(W_{rr}^* + \frac{W_r^*}{r} + \frac{W_{\theta\theta}^*}{r^2} \right) \quad (B-6)$$

$$U_{3b} = (1-\nu) \left[W_{rr}^* \left(\frac{W_r}{r} + \frac{W_{\theta\theta}}{r^2} \right) + W_{rr} \left(\frac{W_r^*}{r} + \frac{W_{\theta\theta}^*}{r^2} \right) \right] \quad (B-7)$$

$$U_{3c} = 2(1-\nu) \left[\frac{W_{r\theta}}{r} - \frac{W_\theta}{r^2} \right] \left[\frac{W_{r\theta}^*}{r} - \frac{W_\theta^*}{r^2} \right] \quad (B-8)$$

A listing of the expansion of the products of the derivatives necessary for the evaluation of U_2 and U_3 are shown in Tables 5 and 6, respectively. Further presentation of the energy integrals is in the notation delineated in those tables.

Equations B-4 and B-5 now become

$$U_2 = \frac{Sh}{4} \sum \sum \sum \sum \int_0^{2\pi} \left[E_1 - E_2 + (E_1 + 3E_3 - 4E_2) \cos 2\theta + E_4 + E_5 - (E_4 + 3E_6) \cos 2\theta - (E_{7a} + E_{7b} - 3E_{9a} - 3E_{9b} + 2E_{8a} + 2E_{8b}) \sin 2\theta \right] d\theta \quad (B-9)$$

and

$$U_3 = \frac{D}{2} \sum \sum \sum \sum \int_0^{2\pi} \left[G_1 + G_{2a} + G_{2b} + G_{3a} + G_{3b} + G_{5a} + G_{5b} + G_4 + G_6 + (1-\nu) (G_{2a} + G_{2b} + G_{3a} + G_{3b}) + 2(1-\nu) (G_7 + G_9 + G_{8a} + G_{8b}) \right] d\theta \quad (B-10)$$

When the integration in θ is done, equations B-9 and B-10 reduce to

$$U_2 = \sum_{I=1}^{\infty} \sum_{J=0}^{\infty} \sum_{K=1}^{\infty} \sum_{L=0}^{\infty} Q_{IJKL} \begin{bmatrix} A_{IJ} \cdot A_{KL} \\ \text{or} \\ B_{IJ} \cdot B_{KL} \end{bmatrix} \quad (B-11)$$

and

$$U_3 = \sum_{I=1}^{\infty} \sum_{J=0}^{\infty} \sum_{K=1}^{\infty} \sum_{L=0}^{\infty} P_{IJKL} \begin{bmatrix} A_{IJ} \cdot A_{KL} \\ \text{or} \\ B_{IJ} \cdot B_{KL} \end{bmatrix} \quad (B-12)$$

Tabulation of the Integrals of the Radius Dependent Terms

NOTATION:

$G_1, G_2, \dots, E_1, E_2, \dots$ etc. are functions of I, J, K, and L.

x means X_{NM} as used previously, only as a function of K & L.

x^* means X_{NM} as used previously, only as a function of I and J.

y means Y_{NM} as used previously, only as a function of K and L.

y^* means Y_{NM} as used previously, only as a function of I and J.

THE INTEGRALS ARE TABULATED ON THE FOLLOWING PAGES. THE FIRST
NINE TERMS IN EACH COLUMN ARE SUMMED TO FORM THE WHOLE EXPRESSION.
THE TERMS $a(J,L)$ or $b(J,L)$ SHOULD BE CHOSEN FOR THE UNKNOWN
COEFFICIENTS A_{NM} or B_{NM} RESPECTIVELY.

Table 5

Tabulation of the Compacted Form of the Terms in the Membrane Energy Integral

$$E_1 = \int_0^{\infty} W_r W_r^* dr = f_1 \cdot \left[\begin{array}{c} a(J,L) \cdot A_{IJ} \cdot A_{KL} \\ \text{or} \\ b(J,L) \cdot B_{IJ} \cdot B_{KL} \end{array} \right]$$

$$E_2 = \int_0^{\infty} \frac{W_r W_r^*}{r^2} dr = f_2 \cdot \left[\begin{array}{c} a(J,L) \cdot A_{IJ} \cdot A_{KL} \\ \text{or} \\ b(J,L) \cdot B_{IJ} \cdot B_{KL} \end{array} \right]$$

$$E_3 = \int_0^{\infty} \frac{W_r W_r^*}{r^4} dr = f_3 \cdot \left[\begin{array}{c} a(J,L) \cdot A_{IJ} \cdot A_{KL} \\ \text{or} \\ b(J,L) \cdot B_{IJ} \cdot B_{KL} \end{array} \right]$$

$$E_4 = \int_0^{\infty} \frac{W_{\theta} W_{\theta}^*}{r^2} dr = f_4 \cdot \left[\begin{array}{c} a(J,L) \cdot A_{IJ} \cdot A_{KL} \\ \text{or} \\ b(J,L) \cdot B_{IJ} \cdot B_{KL} \end{array} \right]$$

$$E_5 = \int_0^{\infty} \frac{W_{\theta} W_{\theta}^*}{r^4} dr = f_5 \cdot \left[\begin{array}{c} a(J,L) \cdot A_{IJ} \cdot A_{KL} \\ \text{or} \\ b(J,L) \cdot B_{IJ} \cdot B_{KL} \end{array} \right]$$

$$E_6 = \int_0^{\infty} \frac{W_{\theta} W_{\theta}^*}{r^6} dr = f_6 \cdot \left[\begin{array}{c} a(J,L) \cdot A_{IJ} \cdot A_{KL} \\ \text{or} \\ b(J,L) \cdot B_{IJ} \cdot B_{KL} \end{array} \right]$$

$$E_{7a} = \int_0^{\infty} \frac{W_{\theta}^* W_r}{r} dr = f_{7a} \cdot \left[\begin{array}{c} a(J,L) \cdot A_{IJ} \cdot A_{KL} \\ \text{or} \\ b(J,L) \cdot B_{IJ} \cdot B_{KL} \end{array} \right]$$

$$E_{7b} = \int_0^{\infty} \frac{W_{\theta} W_r^*}{r} dr = f_{7b} \cdot \left[\begin{array}{c} a(J,L) \cdot A_{IJ} \cdot A_{KL} \\ \text{or} \\ b(J,L) \cdot B_{IJ} \cdot B_{KL} \end{array} \right]$$

Table 5 (continued)

$$E_{8a} = \int_0^{\infty} \frac{W_{\theta}^* W_r}{r^3} dr = f_{8a} \begin{bmatrix} a(J,L) \cdot A_{IJ} \cdot A_{KL} \\ \text{or} \\ b(J,L) \cdot B_{IJ} \cdot B_{KL} \end{bmatrix}$$

$$E_{8b} = \int_0^{\infty} \frac{W_{\theta}^* W_r}{r^3} dr = f_{8b} \begin{bmatrix} a(J,L) \cdot A_{IJ} \cdot A_{KL} \\ \text{or} \\ b(J,L) \cdot B_{IJ} \cdot B_{KL} \end{bmatrix}$$

$$E_{9a} = \int_0^{\infty} \frac{W_{\theta}^* W_r}{r^5} dr = f_{9a} \begin{bmatrix} a(J,L) \cdot A_{IJ} \cdot A_{KL} \\ \text{or} \\ b(J,L) \cdot B_{IJ} \cdot B_{KL} \end{bmatrix}$$

$$E_{9b} = \int_0^{\infty} \frac{W_{\theta}^* W_r}{r^5} dr = f_{9b} \begin{bmatrix} a(J,L) \cdot A_{IJ} \cdot A_{KL} \\ \text{or} \\ b(J,L) \cdot A_{IJ} \cdot A_{KL} \end{bmatrix}$$

Table 5 (continued)

$$\begin{aligned}
f_1 &= \frac{IK}{(I+K)} \\
&+ \frac{I(K+1)x}{(I+K+1)} \\
&+ \frac{I(K+2)y}{(I+K+2)} \\
&+ \frac{(I+1)Kx^*}{(I+K+1)} \\
&+ \frac{(I+1)(K+1)xx^*}{(I+K+2)} \\
&+ \frac{(I+1)(K+2)yx^*}{(I+K+3)} \\
&+ \frac{(I+2)Ky^*}{(I+K+2)} \\
&+ \frac{(I+2)(K+1)y^*x}{(I+K+3)} \\
&+ \frac{(I+2)(K+2)yy^*}{(I+K+4)}
\end{aligned}$$

where

$$a(J,L) = \cos J\theta \cos L\theta$$

$$b(J,L) = \sin J\theta \sin L\theta$$

$$\begin{aligned}
f_2 &= \frac{IK}{(I+K+2)} \\
&+ \frac{I(K+1)x}{(I+K+3)} \\
&+ \frac{I(K+2)y}{(I+K+4)} \\
&+ \frac{(I+1)Kx^*}{(I+K+3)} \\
&+ \frac{(I+1)(K+1)xx^*}{(I+K+4)} \\
&+ \frac{(I+1)(K+2)yx^*}{(I+K+5)} \\
&+ \frac{(I+2)Ky^*}{(I+K+4)} \\
&+ \frac{(I+2)(K+1)xy^*}{(I+K+5)} \\
&+ \frac{(I+2)(K+2)yy^*}{(I+K+6)}
\end{aligned}$$

where

$$a(J,L) = \cos J\theta \cos L\theta$$

$$b(J,L) = \sin J\theta \sin L\theta$$

Table 5 (continued)

$$\begin{aligned}
f_3 &= \frac{IK}{(I+K+4)} \\
&+ \frac{I(K+1)x}{(I+K+5)} \\
&+ \frac{I(K+2)y}{(I+K+6)} \\
&+ \frac{(I+1)Kx^*}{(I+K+5)} \\
&+ \frac{(I+1)(K+1)xx^*}{(I+K+6)} \\
&+ \frac{(I+1)(K+2)yx^*}{(I+K+7)} \\
&+ \frac{(I+2)Ky^*}{(I+K+6)} \\
&+ \frac{(I+2)(K+1)xy^*}{(I+K+7)} \\
&+ \frac{(I+2)(K+2)yy^*}{(I+K+8)} \\
f_4 &= \frac{JL}{(I+K)} \\
&+ \frac{JLx^*}{(I+K+1)} \\
&+ \frac{JLy^*}{(I+K+2)} \\
&+ \frac{JLx}{(I+K+1)} \\
&+ \frac{JLxx^*}{(I+K+2)} \\
&+ \frac{JLxy^*}{(I+K+3)} \\
&+ \frac{JLy}{(I+K+2)} \\
&+ \frac{JLyx^*}{(I+K+3)} \\
&+ \frac{JLyy^*}{(I+K+4)}
\end{aligned}$$

where

$$a(J,L) = \cos J\theta \cos L\theta$$

$$b(J,L) = \sin J\theta \sin L\theta$$

where

$$a(J,L) = \sin J\theta \sin L\theta$$

$$b(J,L) = \cos J\theta \cos L\theta$$

Table 5 (continued)

$$f_5 = \frac{JL}{(I+K+2)}$$

$$+ \frac{JLx^*}{(I+K+3)}$$

$$+ \frac{JLy^*}{(I+K+4)}$$

$$+ \frac{JLx}{(I+K+3)}$$

$$+ \frac{JLxx^*}{(I+K+4)}$$

$$+ \frac{JLxy^*}{(I+K+5)}$$

$$+ \frac{JLy}{(I+K+4)}$$

$$+ \frac{JLyx^*}{(I+K+5)}$$

$$+ \frac{JLyy^*}{(I+K+6)}$$

$$f_6 = \frac{JL}{(I+K+4)}$$

$$+ \frac{JLx^*}{(I+K+5)}$$

$$+ \frac{JLy^*}{(I+K+6)}$$

$$+ \frac{JLx}{(I+K+5)}$$

$$+ \frac{JLxx^*}{(I+K+6)}$$

$$+ \frac{JLxy^*}{(I+K+7)}$$

$$+ \frac{JLy}{(I+K+6)}$$

$$+ \frac{JLyx^*}{(I+K+7)}$$

$$+ \frac{JLyy^*}{(I+K+8)}$$

where

$$a(J,L) = \sin J\theta \sin L\theta$$

$$b(J,L) = \cos J\theta \cos L\theta$$

where

$$a(J,L) = \sin J\theta \sin L\theta$$

$$b(J,L) = \cos J\theta \cos L\theta$$

Table 5 (continued)

$$f_{7a} = - \frac{JK}{(I+K)}$$

$$- \frac{J(K+1)x}{(I+K+1)}$$

$$- \frac{J(K+2)y}{(I+K+2)}$$

$$- \frac{JKx^*}{(I+K+1)}$$

$$- \frac{J(K+1)xx^*}{(I+K+2)}$$

$$- \frac{J(K+2)yx^*}{(I+K+3)}$$

$$- \frac{JKy^*}{(I+K+2)}$$

$$- \frac{J(K+1)xy^*}{(I+K+3)}$$

$$- \frac{J(K+2)yy^*}{(I+K+4)}$$

where

$$a(J,L) = -\sin J\theta \cos L\theta$$

$$b(J,L) = \sin L\theta \cos J\theta$$

$$f_{7b} = - \frac{LI}{(I+K)}$$

$$- \frac{L(I+1)x^*}{(I+K+1)}$$

$$- \frac{L(I+2)y^*}{(I+K+2)}$$

$$- \frac{LIx}{(I+K+1)}$$

$$- \frac{L(I+1)xx^*}{(I+K+2)}$$

$$- \frac{L(I+2)xy^*}{(I+K+3)}$$

$$- \frac{LIy}{(I+K+2)}$$

$$- \frac{L(I+1)yx^*}{(I+K+3)}$$

$$- \frac{L(I+2)yy^*}{(I+K+4)}$$

where

$$a(J,L) = -\sin L\theta \cos J\theta$$

$$b(J,L) = \sin J\theta \cos L\theta$$

Table 5 (continued)

$$f_{8a} = - \frac{JK}{(I+K+2)}$$

$$- \frac{J(K+1)x}{(I+K+3)}$$

$$- \frac{J(K+2)y}{(I+K+4)}$$

$$- \frac{JKx^{**}}{(I+K+3)}$$

$$- \frac{J(K+1)xx^{**}}{(I+K+4)}$$

$$- \frac{J(K+2)yx^{**}}{(I+K+5)}$$

$$- \frac{JKy^{**}}{(I+K+4)}$$

$$- \frac{J(K+1)xy^{**}}{(I+K+5)}$$

$$- \frac{J(K+2)yy^{**}}{(I+K+6)}$$

$$f_{8b} = - \frac{LI}{(I+K+2)}$$

$$- \frac{L(I+1)x^{**}}{(I+K+3)}$$

$$- \frac{L(I+2)y^{**}}{(I+K+4)}$$

$$- \frac{LIx}{(I+K+3)}$$

$$- \frac{L(I+1)xx^{**}}{(I+K+4)}$$

$$- \frac{L(I+2)xy^{**}}{(I+K+5)}$$

$$- \frac{LIy}{(I+K+4)}$$

$$- \frac{L(I+1)yx^{**}}{(I+K+5)}$$

$$- \frac{L(I+2)yy^{**}}{(I+K+6)}$$

where

$$a(J,L) = -\sin J\theta \cos L\theta$$

$$b(J,L) = \sin L\theta \cos J\theta$$

where

$$a(J,L) = -\sin L\theta \cos J\theta$$

$$b(J,L) = \sin J\theta \cos L\theta$$

Table 5 (continued)

$$f_{9a} = - \frac{JK}{(I+K+4)}$$

$$- \frac{J(K+1)x}{(I+K+5)}$$

$$- \frac{J(K+2)y}{(I+K+6)}$$

$$- \frac{JKx^*}{(I+K+5)}$$

$$- \frac{J(K+1)xx^*}{(I+K+6)}$$

$$- \frac{J(K+2)yx^*}{(I+K+7)}$$

$$- \frac{JKy^*}{(I+K+6)}$$

$$- \frac{J(K+1)xy^*}{(I+K+7)}$$

$$- \frac{J(K+2)yy^*}{(I+K+8)}$$

$$f_{9b} = - \frac{LI}{(I+K+4)}$$

$$- \frac{L(I+1)x^*}{(I+K+5)}$$

$$- \frac{L(I+2)y^*}{(I+K+6)}$$

$$- \frac{LIx}{(I+K+5)}$$

$$- \frac{L(I+1)xx^*}{(I+K+6)}$$

$$- \frac{L(I+2)xy^*}{(I+K+7)}$$

$$- \frac{LIy}{(I+K+6)}$$

$$- \frac{L(I+1)yx^*}{(I+K+7)}$$

$$- \frac{L(I+2)yy^*}{(I+K+8)}$$

where

$$a(J,L) = -\sin J\theta \cos L\theta$$

$$b(J,L) = \sin L\theta \cos J\theta$$

where

$$a(J,L) = -\sin L\theta \cos J\theta$$

$$b(J,L) = \sin J\theta \cos L\theta$$

Table 6

Tabulation of the Compacted Form of the Terms in
the Bending Energy Integral

$$G_1 = \int_0^\infty W_{rr} W_{rr}^* dr = g_1 \left[\begin{array}{c} a(J,L) \cdot A_{IJ} \cdot A_{KL} \\ \text{or} \\ b(J,L) \cdot B_{IJ} \cdot B_{KL} \end{array} \right]$$

$$G_{2a} = \int_0^\infty \frac{W_r^* W_{rr}}{r} dr = g_{2a} \left[\begin{array}{c} a(J,L) \cdot A_{IJ} \cdot A_{KL} \\ \text{or} \\ b(J,L) \cdot B_{IJ} \cdot B_{KL} \end{array} \right]$$

$$G_{2b} = \int_0^\infty \frac{W_r W_{rr}^*}{r} dr = g_{2b} \left[\begin{array}{c} a(J,L) \cdot A_{IJ} \cdot A_{KL} \\ \text{or} \\ b(J,L) \cdot B_{IJ} \cdot B_{KL} \end{array} \right]$$

$$G_{3a} = \int_0^\infty \frac{W_{rr}^* W_{\theta\theta}}{r^2} dr = g_{3a} \left[\begin{array}{c} a(J,L) \cdot A_{IJ} \cdot A_{KL} \\ \text{or} \\ b(J,L) \cdot B_{IJ} \cdot B_{KL} \end{array} \right]$$

$$G_{3b} = \int_0^\infty \frac{W_{rr} W_{\theta\theta}^*}{r^2} dr = g_{3b} \left[\begin{array}{c} a(J,L) \cdot A_{IJ} \cdot A_{KL} \\ \text{or} \\ b(J,L) \cdot B_{IJ} \cdot B_{KL} \end{array} \right]$$

$$G_4 = \int_0^\infty \frac{W_r W_r^*}{r^2} dr = g_4 \left[\begin{array}{c} a(J,L) \cdot A_{IJ} \cdot A_{KL} \\ \text{or} \\ b(J,L) \cdot B_{IJ} \cdot B_{KL} \end{array} \right]$$

$$G_{5a} = \int_0^\infty \frac{W_{\theta\theta}^* W_r}{r^3} dr = g_{5a} \left[\begin{array}{c} a(J,L) \cdot A_{IJ} \cdot A_{KL} \\ \text{or} \\ b(J,L) \cdot B_{IJ} \cdot B_{KL} \end{array} \right]$$

$$G_{5b} = \int_0^\infty \frac{W_{\theta\theta} W_r^*}{r^3} dr = g_{5b} \left[\begin{array}{c} a(J,L) \cdot A_{IJ} \cdot A_{KL} \\ \text{or} \\ b(J,L) \cdot B_{IJ} \cdot B_{KL} \end{array} \right]$$

Table 6 (continued)

$$G_6 = \int_0^{\infty} \frac{W_{\theta\theta} W_{\theta\theta}^*}{r^4} dr = g_6 \cdot \begin{bmatrix} a(J,L) \cdot A_{IJ} \cdot A_{KL} \\ \text{or} \\ b(J,L) \cdot B_{IJ} \cdot B_{KL} \end{bmatrix}$$

$$G_7 = \int_0^{\infty} \frac{W_{re} W_{re}^*}{r^2} dr = g_7 \cdot \begin{bmatrix} a(J,L) \cdot A_{IJ} \cdot A_{KL} \\ \text{or} \\ b(J,L) \cdot B_{IJ} \cdot B_{KL} \end{bmatrix}$$

$$G_{8a} = \int_0^{\infty} \frac{W_{\theta}^* W_{re}}{r^3} dr = g_{8a} \cdot \begin{bmatrix} a(J,L) \cdot A_{IJ} \cdot A_{KL} \\ \text{or} \\ b(J,L) \cdot B_{IJ} \cdot B_{KL} \end{bmatrix}$$

$$G_{8b} = \int_0^{\infty} \frac{W_{\theta} W_{re}^*}{r^3} dr = g_{8b} \cdot \begin{bmatrix} a(J,L) \cdot A_{IJ} \cdot A_{KL} \\ \text{or} \\ b(J,L) \cdot B_{IJ} \cdot B_{KL} \end{bmatrix}$$

$$G_9 = \int_0^{\infty} \frac{W_{\theta} W_{\theta}^*}{r^4} dr = g_9 \cdot \begin{bmatrix} a(J,L) \cdot A_{IJ} \cdot A_{KL} \\ \text{or} \\ b(J,L) \cdot B_{IJ} \cdot B_{KL} \end{bmatrix}$$

Table 6 (continued)

$$\begin{aligned}
 g_1 = & \frac{IK(I+1)(K+1)}{(I+K+2)} \\
 & + \frac{(I+1)(K+1)(I+2)(K+2)xx^*}{(I+K+4)} \\
 & + \frac{(I+2)(K+2)(I+3)(K+3)yy^*}{(I+K+6)} \\
 & + \frac{K(K+1)(I+1)(I+2)x^*}{(I+K+3)} \\
 & + \frac{I(I+1)(K+1)(K+2)x}{(I+K+3)} \\
 & + \frac{K(K+1)(I+2)(I+3)y^*}{(I+K+4)} \\
 & + \frac{I(I+1)(K+2)(K+3)y}{(I+K+4)} \\
 & + \frac{(I+1)(I+2)(K+2)(K+3)yx^*}{(I+K+5)} \\
 & + \frac{(K+1)(K+2)(I+2)(I+3)xy^*}{(I+K+5)}
 \end{aligned}$$

where

$$a(J,L) = \cos J\theta \cos L\theta$$

$$b(J,L) = \sin J\theta \sin L\theta$$

Table 6 (continued)

$$g_{2a} = - \frac{IK(K+1)}{(I+K+2)}$$

$$- \frac{I(K+1)(K+2)x}{(I+K+3)}$$

$$- \frac{I(K+2)(K+3)y}{(I+K+4)}$$

$$- \frac{(I+1)(K+1)Kx^{**}}{(I+K+3)}$$

$$- \frac{(I+1)(K+1)(K+2)xx^{**}}{(I+K+4)}$$

$$- \frac{(I+1)(K+2)(K+3)yx^{**}}{(I+K+5)}$$

$$- \frac{(I+2)(K+1)Ky^{**}}{(I+K+4)}$$

$$- \frac{(I+2)(K+1)(K+2)xy^{**}}{(I+K+5)}$$

$$- \frac{(I+2)(K+2)(K+3)yy^{**}}{(I+K+6)}$$

$$g_{2b} = - \frac{IK(I+1)}{(I+K+2)}$$

$$- \frac{K(I+1)(I+2)x^{**}}{(I+K+3)}$$

$$- \frac{K(I+2)(I+3)y^{**}}{(I+K+4)}$$

$$- \frac{(K+1)I(I+1)x}{(I+K+3)}$$

$$- \frac{(K+1)(I+1)(I+2)xx^{**}}{(I+K+4)}$$

$$- \frac{(K+1)(I+2)(I+3)xy^{**}}{(I+K+4)}$$

$$- \frac{(K+2)I(I+1)y}{(I+K+4)}$$

$$- \frac{(K+2)(I+1)(I+2)yx^{**}}{(I+K+5)}$$

$$- \frac{(K+2)(I+2)(I+3)yy^{**}}{(I+K+6)}$$

where

$$a(J,L) = \cos J\theta \cos L\theta$$

$$b(J,L) = \sin J\theta \sin L\theta$$

where

$$a(J,L) = \cos J\theta \cos L\theta$$

$$b(J,L) = \sin J\theta \sin L\theta$$

Table 6 (continued)

$$\begin{aligned}
g_{3a} = & - \frac{I(I+1)L^2}{(I+K+2)} \\
& - \frac{I(I+1)L^2_x}{(I+K+3)} \\
& - \frac{I(I+1)L^2_y}{(I+K+4)} \\
& - \frac{(I+1)(I+2)L^2_x^*}{(I+K+3)} \\
& - \frac{(I+1)(I+2)L^2_{xx}^*}{(I+K+4)} \\
& - \frac{(I+1)(I+2)L^2_{yx}^*}{(I+K+5)} \\
& - \frac{(I+2)(I+3)L^2_y^*}{(I+K+4)} \\
& - \frac{(I+2)(I+3)L^2_{xy}^*}{(I+K+5)} \\
& - \frac{(I+2)(I+3)L^2_{yy}^*}{(I+K+6)}
\end{aligned}$$

where

$$a(J,L) = \cos J\theta \cos L\theta$$

$$b(J,L) = \sin J\theta \sin L\theta$$

$$\begin{aligned}
g_{3b} = & - \frac{K(K+1)J^2}{(I+K+2)} \\
& - \frac{K(K+1)J^2_x^*}{(I+K+3)} \\
& - \frac{K(K+1)J^2_y^*}{(I+K+4)} \\
& - \frac{(K+1)(K+2)J^2_x}{(I+K+3)} \\
& - \frac{(K+1)(K+2)J^2_{xx}^*}{(I+K+4)} \\
& - \frac{(K+1)(K+2)J^2_{xy}^*}{(I+K+5)} \\
& - \frac{(K+2)(K+3)J^2_y}{(I+K+4)} \\
& - \frac{(K+2)(K+3)J^2_{yx}^*}{(I+K+5)} \\
& - \frac{(K+2)(K+3)J^2_{yy}^*}{(I+K+6)}
\end{aligned}$$

where

$$a(J,L) = \cos J\theta \cos L\theta$$

$$b(J,L) = \sin J\theta \sin L\theta$$

Table 6 (continued)

$$\begin{aligned}
g_4 = & \frac{IK}{(I+K+2)} \\
& + \frac{K(I+1)x^*}{(I+K+3)} \\
& + \frac{K(I+2)y^*}{(I+K+4)} \\
& + \frac{(K+1)Ix}{(I+K+3)} \\
& + \frac{(K+1)(I+1)xx^*}{(I+K+4)} \\
& + \frac{(K+1)(I+2)xy^*}{(I+K+5)} \\
& + \frac{(K+2)Iy}{(I+K+4)} \\
& + \frac{(K+2)(I+1)yx^*}{(I+K+5)} \\
& + \frac{(K+2)(I+2)yy^*}{(I+K+6)}
\end{aligned}$$

where

$$a(J,L) = \cos J\theta \cos L\theta$$

$$b(J,L) = \sin J\theta \sin L\theta$$

$$\begin{aligned}
g_{5a} = & \frac{J^2 L}{(I+K+2)} \\
& + \frac{J^2 (K+1)x}{(I+K+3)} \\
& + \frac{J^2 (K+2)y}{(I+K+4)} \\
& + \frac{J^2 Kx^*}{(I+K+3)} \\
& + \frac{J^2 (K+1)xx^*}{(I+K+4)} \\
& + \frac{J^2 (K+2)yx^*}{(I+K+5)} \\
& + \frac{J^2 Ky^*}{(I+K+4)} \\
& + \frac{J^2 (K+1)xy^*}{(I+K+5)} \\
& + \frac{J^2 (K+2)yy^*}{(I+K+6)}
\end{aligned}$$

where

$$a(J,L) = \cos J\theta \cos L\theta$$

$$b(J,L) = \sin J\theta \sin L\theta$$

Table 6 (continued)

$$\begin{aligned}
g_{5b} = & \frac{L^2 I}{(I+K+2)} \\
& + \frac{L^2 (I+1) x^*}{(I+K+3)} \\
& + \frac{L^2 (I+2) y^*}{(I+K+4)} \\
& + \frac{L^2 I x}{(I+K+3)} \\
& + \frac{L^2 (I+1) x x^*}{(I+K+4)} \\
& + \frac{L^2 (I+2) x y^*}{(I+K+5)} \\
& + \frac{L^2 I y}{(I+K+4)} \\
& + \frac{L^2 (I+1) y x^*}{(I+K+5)} \\
& + \frac{L^2 (I+2) y y^*}{(I+K+6)}
\end{aligned}$$

where

$$a(J, L) = \cos J\theta \cos L\theta$$

$$b(J, L) = \sin J\theta \sin L\theta$$

$$\begin{aligned}
g_6 = & \frac{J^2 L^2}{(I+K+2)} \\
& + \frac{J^2 L^2 x^*}{(I+K+3)} \\
& + \frac{J^2 L^2 y^*}{(I+K+4)} \\
& + \frac{J^2 L^2 x^*}{(I+K+3)} \\
& + \frac{J^2 L^2 x x^*}{(I+K+4)} \\
& + \frac{J^2 L^2 x y^*}{(I+K+5)} \\
& + \frac{J^2 L^2 y}{(I+K+4)} \\
& + \frac{J^2 L^2 y x^*}{(I+K+5)} \\
& + \frac{J^2 L^2 y y^*}{(I+K+6)}
\end{aligned}$$

where

$$a(J, L) = \cos J\theta \cos L\theta$$

$$b(J, L) = \sin J\theta \sin L\theta$$

Table 6 (continued)

$$\begin{aligned}
 g_7 &= \frac{IJKL}{(I+K+2)} \\
 &+ \frac{IJ(K+1)Lx}{(I+K+3)} \\
 &+ \frac{IJ(K+2)Ly}{(I+K+4)} \\
 &+ \frac{(I+1)JKLx^*}{(I+K+3)} \\
 &+ \frac{(I+1)J(K+1)Lxx^*}{(I+K+4)} \\
 &+ \frac{(I+1)J(K+2)Lyx^*}{(I+K+5)} \\
 &+ \frac{(I+2)JKLy^*}{(I+K+4)} \\
 &+ \frac{(I+2)J(K+1)Lxy^*}{(I+K+5)} \\
 &+ \frac{(I+2)J(K+2)Lyy^*}{(I+K+6)}
 \end{aligned}$$

where

$$a(J,L) = \sin J\theta \sin L\theta$$

$$b(J,L) = \cos J\theta \cos L\theta$$

$$\begin{aligned}
 g_{8a} &= - \frac{JK}{(I+K+2)} \\
 &- \frac{J(K+1)x}{(I+K+3)} \\
 &- \frac{J(K+2)y}{(I+K+4)} \\
 &- \frac{JKx^*}{(I+K+3)} \\
 &- \frac{J(K+1)xx^*}{(I+K+4)} \\
 &- \frac{J(K+2)yx^*}{(I+K+5)} \\
 &- \frac{JKy^*}{(I+K+4)} \\
 &- \frac{J(K+1)xy^*}{(I+K+5)} \\
 &- \frac{J(K+2)yy^*}{(I+K+6)}
 \end{aligned}$$

where

$$a(J,L) = \sin J\theta \sin L\theta$$

$$b(J,L) = \cos J\theta \cos L\theta$$

Table 6 (continued)

$$\begin{aligned}
g_{8a} = & - \frac{LI}{(I+K+2)} & g_9 = & \frac{JL}{(I+K+2)} \\
& - \frac{L(I+1)x^*}{(I+K+3)} & & + \frac{JLx^*}{(I+K+3)} \\
& - \frac{L(I+2)y^*}{(I+K+4)} & & + \frac{JLy^*}{(I+K+4)} \\
& - \frac{LIx}{(I+K+3)} & & + \frac{JLx}{(I+K+3)} \\
& - \frac{L(I+1)xx^*}{(I+K+3)} & & + \frac{JLxx^*}{(I+K+4)} \\
& - \frac{L(I+2)xy^*}{(I+K+4)} & & + \frac{JLxy^*}{(I+K+5)} \\
& - \frac{LIy}{(I+K+4)} & & + \frac{JLy}{(I+K+4)} \\
& - \frac{L(I+1)yx^*}{(I+K+5)} & & + \frac{JLyx^*}{(I+K+5)} \\
& - \frac{L(I+2)yy^*}{(I+K+6)} & & + \frac{JLyy^*}{(I+K+6)}
\end{aligned}$$

where

$$a(J,L) = \sin J\theta \sin L\theta$$

$$b(J,L) = \cos J\theta \cos L\theta$$

where

$$a(J,L) = \sin J\theta \sin L\theta$$

$$b(J,L) = \cos J\theta \cos L\theta$$

APPENDIX C

INTEGRATION OF SINE-COSINE TRIPLE PRODUCTS

When evaluating the energy integrals for the plate, it is necessary to evaluate the following two integrals

$$I_c = \int_0^{2\pi} (\cos A\theta \cos B\theta \cos C\theta) d\theta \quad (C-1)$$

and

$$I_s = \int_0^{2\pi} (\sin A\theta \sin B\theta \cos C\theta) d\theta \quad (C-2)$$

where A, B, and C are integers.

The above integrals can best be evaluated by expanding the integrands using the identities

$$\sin x \cos y = \frac{1}{2} [\sin(x+y) + \sin(x-y)] \quad (C-3)$$

$$\cos x \sin y = \frac{1}{2} [\sin(x+y) - \sin(x-y)] \quad (C-4)$$

$$\cos x \cos y = \frac{1}{2} [\cos(x+y) + \cos(x-y)] \quad (C-5)$$

$$\sin x \sin y = \frac{1}{2} [\cos(x-y) - \cos(x+y)] \quad (C-6)$$

The object is to reduce the integrals to the form

$$\int_0^{2\pi} \sin D\theta d\theta \quad \text{or} \quad \int_0^{2\pi} \cos D\theta d\theta \quad (C-7)$$

where D is an integer sum or difference of A, B, and C. Note that the sine integral in C-7 is always zero and that the cosine integral is nonzero only if D is zero.

Expanding equation C-2, using identity C-3, we get

$$I_s = \frac{1}{2} \int_0^{2\pi} \sin A\theta [\sin(B+C)\theta + \sin(B-C)\theta] d\theta \quad (C-8)$$

Continuing the process with identity C-6

$$I_s = \frac{1}{4} \int_0^{2\pi} [\cos(A-B-C)\theta - \cos(A+B+C)\theta + \cos(A-B+C)\theta - \cos(A+B-C)\theta] d\theta \quad (C-9)$$

I_s is now expressed as the sum of four integrals

$$I_s = I_1 + I_2 + I_3 + I_4 \quad (C-10)$$

where

$$I_1 = \begin{cases} \pi/2 & \text{if } (A-B-C) = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$I_2 = \begin{cases} -\pi/2 & \text{if } (A+B+C) = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$I_3 = \begin{cases} \pi/2 & \text{if } (A-B+C) = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$I_4 = \begin{cases} -\pi/2 & \text{if } (A+B-C) = 0 \\ 0 & \text{otherwise} \end{cases}$$

A similar expansion for I_c (equation C-1) can be made. The results are the same as above, except that the signs of I_2 and I_4 are positive.

APPENDIX D

SOLUTION OF THE EIGENVALUE PROBLEM $(\underline{\underline{R}}\lambda + \underline{\underline{T}}) \underline{\underline{A}} = 0$

The canonical form of the matrix eigenvalue problem is

$$(\underline{\underline{I}}\lambda - \underline{\underline{B}}) \underline{\underline{X}} = 0 \quad (\text{D-1})$$

where $\underline{\underline{I}}$ is the identity matrix, $\underline{\underline{X}}$ an unknown eigenvector, λ is an unknown scalar parameter (eigenvalue), and $\underline{\underline{B}}$ a known coefficient matrix.

Equation D-1 has a solution if determinant of

$$(\underline{\underline{I}}\lambda - \underline{\underline{B}}) = 0 \quad (\text{D-2})$$

If this determinant is expanded, a polynomial equation in λ results. However, if the determinant is of order greater than three or four, it is generally inefficient to obtain this polynomial equation and then solve it.

There are alternative procedures for solution of the eigenvalue problem. A method which gives extremely accurate eigenvectors and eigenvalues for symmetric matrices is one named for Jacobi, which is presented by Fox (1). Brown and White (13) give a comparison of the accuracy of the Jacobi method with that of other methods available. The necessity for extremely accurate eigenvectors is shown later.

Consider the eigenvalue problem (equation 33)

$$(\underline{\underline{R}}\lambda + \underline{\underline{T}}) \underline{\underline{A}} = 0 \quad (\text{D-3})$$

where both $\underline{\underline{R}}$ and $\underline{\underline{T}}$ are symmetric matrices and $\underline{\underline{T}}$ is positive definite. Dividing equation D-3 by λ , and making the substitution

$$y = -\frac{1}{\lambda} \quad (\text{D-4})$$

gives

$$(\underline{\underline{T}}y - \underline{\underline{R}}) \underline{\underline{A}} = 0 \quad (\text{D-5})$$

Any symmetric matrix with real elements can be reduced to a product of the eigenvector and eigenvalue matrices, so that

$$\underline{T} = \underline{Q} \underline{D} \underline{Q}^T \quad (D-6)$$

where $\underline{D}$ is a diagonal matrix of the eigenvalues (d_i) of $\underline{T}$, and $\underline{Q}$ is a matrix of normalized eigenvectors (q_i), obtained from a solution of the problem

$$(\underline{T} d_i - \underline{T}) \underline{q}_i = 0 \quad (D-7)$$

Note that $\underline{Q}^T = \underline{Q}^{-1}$ since the matrix $\underline{T}$ is symmetric and has orthogonal eigenvectors.

Equation D-5 may now be written

$$(\underline{Q} \underline{D} \underline{Q}^T \underline{y} - \underline{R}) \underline{A} = 0 \quad (D-8)$$

Premultiplying by $\underline{Q}^T$, we get

$$(\underline{D} \underline{Q}^T \underline{y} - \underline{Q}^T \underline{R}) \underline{A} = 0 \quad (D-9)$$

Factoring a $\underline{Q}$ out of $\underline{A} = \underline{I} \underline{A} = \underline{Q} \underline{Q}^T \underline{A}$, and letting

$$\underline{Y} = \underline{Q}^T \underline{A} \quad (D-10)$$

the problem reduces to

$$(\underline{D} \underline{y} - \underline{Q}^T \underline{R} \underline{Q}) \underline{Y} = 0 \quad (D-11)$$

Since $\underline{T}$ is positive definite, $\underline{D}$ contains all positive terms. By again premultiplying, this time by $\underline{D}^{-1/2}$, factoring $\underline{D}^{-1/2}$, and making the substitution

$$\underline{Z} = \underline{D}^{-1/2} \underline{Y} \quad (D-12)$$

equation D-11 reduces to

$$(\underline{I} \underline{y} - \underline{D}^{-1/2} \underline{Q}^T \underline{R} \underline{Q} \underline{D}^{-1/2}) \underline{Z} = 0 \quad (D-13)$$

and becomes

$$(\underline{\underline{I}}\gamma - \underline{\underline{B}})\underline{\underline{Z}} = 0 \quad (\text{D-14})$$

which is of canonical form.

Note that $\underline{\underline{D}}^{\frac{1}{2}}$ and $\underline{\underline{D}}^{-\frac{1}{2}}$ are easily obtained by taking the square roots, and the square root reciprocals, respectively, of the diagonal terms. Also note that $\underline{\underline{B}}$ is symmetric. The problem is now well suited for solution by use of the Jacobi method. It is seen that inaccuracies in obtaining eigenvectors $\underline{\underline{Q}}$ would cause compounded errors in the final solution for $\underline{\underline{A}}$ and λ .

The problem is returned to original form with the substitutions

$$\lambda = -\frac{1}{\gamma} \quad (\text{D-15})$$

and

$$\underline{\underline{A}} = \underline{\underline{Q}}\underline{\underline{D}}^{-1/2}\underline{\underline{Z}} \quad (\text{D-16})$$

APPENDIX E

SUGGESTIONS FOR EXTENSION OF THE PROBLEM

There are two obvious directions for extension of this problem of a thin plate with a circular hole. The buckling problem for a strip of finite width and the postbuckling problem for both finite and infinite plates remain to be treated analytically.

The postbuckling problem is the more difficult of the two because of the coupling between the stress distribution and the buckled deflection. Johns (2) outlines the postbuckling situation for plates and suggests several methods of solution. Solution of the postbuckling problem would be important because of the revised stress concentration factors that would be obtained; cf. Houghton and Rothwell (16).

Extension of this problem to a finite strip, of width $2b$, would require the following:

- a. Using the Howland solution (17) instead of that due to Kirsch, to obtain the stress distribution in the plate.
- b. Satisfying additional boundary conditions for two "free" straight edges.

Integration of the energy terms for the finite strip is between the limits $r = a$ and $r = \frac{b}{\sin \theta}$, so that the integration with respect to θ becomes more complicated than for the infinite plate.

This integration is over the interval 0 to 2π . Wylie (10) presents a method for the evaluation of real trigonometric integrals over this interval. The method involves substitution of appropriate complex variables for the sine and cosine terms, and then evaluation of the residues inside a unit circle.

There are two possible methods for satisfaction of the additional boundary conditions of the finite strip. Introduction of additional unknowns, similar to X_{NM} and Y_{NM} in equation A-3, into the assumed series for deflection could be made. These unknowns could be determined by simultaneous solution of all the boundary condition equations. Another method, the Lagrangian multiplier method, introduces dummy variables so that additional restraining equations, those expressing the boundary conditions, can be solved simultaneously with the energy expressions. Budiansky and Hu (12) present a clear example of the use of this method in minimizing the plate energy, while satisfying plate boundary conditions.

APPENDIX F
COMPUTER PROGRAM LISTINGS

The following pages contain a listing of the computer programs used in the solution of the problem. All programs are written in FORTRAN IV for use on an IBM System 360 computer.

The control card listing on the first page is effective as of September 1967. Users of this program should check current procedures.

The maximum terms to be taken in N and M, and the plate thickness read into the computer by data card (see comment cards regarding input requirements). Young's modulus, E, and Poisson's ratio are established by cards serial 0010 and 0011 in the main program.

Printed output from the computer is a column matrix EIVU which is an array of all of the values of S_{cr} in psi for the parameters of plate thickness, Poisson's ratio, Young's modulus, terms in N, and terms in M. Matrix EIVU is followed by a map of the buckled plate deflection for the minimum value of tensile critical buckling stress.

```

//LBJECT JCB '400', 'PELLET', MSGLEVEL=1
//EXEC FORTCLG
//FCKT EXEC PCN=IEYFCKT
//SYSLIN CC DSN=IEYFCKT
//SYSFRINT DD DSN=SYS1.SYSUT4, DISP=OLD
//SYSFRACH DD SYSCUT=A
//FCKT.SYSIN DD UNIT=SYSCP
//          DD *
00000010
00000020
00000030
00000040

```

The above card listing is the initial control card sequence needed for the IBM System 360 as of September, 1967. Users of this program should insure that the above cards are valid before using them. Only the unnumbered cards need to be supplied by the user.

0001

IMPLICIT REAL*(A-H,O-Z)
THIS IS THE MAIN PROGRAM FOR DETERMINING THE CRITICAL BUCKLING
STRESS FOR AN INFINITE PLATE UNDER TENSION.

SEE - PELLET, C. A. 'BUCKLING OF AN INFINITE PLATE WITH A HOLE
UNDER TENSION, NPGS MASTERS THESIS, 1967

THIS PROGRAM FINDS THE ENERGY STORED IN AN INFINITE PLATE WITH
A HOLE, AND MINIMIZES THIS ENERGY WITH RESPECT TO AN UNKNOWN
PARAMETER. CRITICAL BUCKLING STRESS IS PRINTED IN THE MATRIX
EIVU. THE UNKNOWN COEFFICIENTS ARE IN MATRIX A. BUCKLED MODE
SHAPES FOR THE PLATE ARE PRINTED.

INPUT REQUIREMENT - DATA CARDS AS SPECIFIED BY FORMATS 991 & 930
OUTPUT - PRINTED OUTPUT OF EIVU, A, AND BUCKLED MODE SHAPES

COMMON/PASS/MCOUNT(100),JOY,N,M,INIT,THICK
COMMON/PAST/JRCW,JCOL
COMMON/HANE/XX(100),YY(100)
COMMON/CHUNK/A(80,80),H(80,80),EIVR(80,80)
COMMON/MUCK/EIVU(100),TEMPA(100)
COMMON/TAME/START

COMMON/NUMBER/LS1,US2,US3,US4,US5,US6,US7A,US7B,US8A,US8B,US9A,
1 US9B,UB1,UB2A,UB2B,UB3A,UB3B,UB4,UB5A,UB5B,UB7,UB8A,UB8B,UB9,UB6,
2 UBD,USD

START=ITIME(C)*.C1

E=30.D+06

VNU=.3D0

READ 991,NCARD

FORMAT(12)

DC 84 JARK=1,NCARD

READ 5,930) N,M,THICK

RIGID=E*THICK**3/(12.*(1.-VNU**2))

FORMAT(213,CF.7)

INIT=N*M

GENERATE ODD OR EVEN M'S,WITH SIN OR COS TERMS

DC 844 JOY=1,3,2

DC 364 ITLE=1,M

GC TC (365,366,367,368),JOY

MCOUNT(ITLE)=2*ITL-1

GC TC 364

MCOUNT(ITLE)=2*ITL-2

GC TC 364

MCOUNT(ITLE)=2*ITL-1

GC TC 364

MCOUNT(ITLE)=2*ITL

CONTINUE

```

0030 DO 363 NOD=1,INIT
0031 DG 363 NUT=1,INIT
0032 H(NUT,NOD)=C.DO
0033 A(NUT,NOD)=C.DO
0034 NEWPAG =50
0035 PIE=3.141592653589793238
0036 DO 30 I=1,N
0037 DO 30 J=1,M
0038 JCCL=(I-1)*M+J
0039 CALL BOUND(I,MCCOUNT(J),XX(JCOL),YY(JCOL))
0040 DO 800 I=1,N
0041 DO 800 JAA=1,M
0042 DO 800 K=1,N
0043 DO 800 LAA=1,M
0044 J=MCCOUNT(JAA)
0045 L=MCCOUNT(LAA)
C CCNVERT QUADRUPLER SUBSCRIPTS TO DOUBLE SUBSCRIPTS.
0046 JCCL=(K-1)*M+LAA
0047 JROW=(I-1)*M+JAA
C INITIALIZE THE VALUE OF THE TRIGONOMETRIC INTEGRALS
C NOTATION IS OF THE FORM CACBCD. THIS MEANS THE INTEGRAL OF
C COSINE (A*THETA) COSINE (B*THETA) OVER THE
C INTERVAL ZERO TO TWO PI
0048 CLCJ=0.DO
0049 SLSJ=0.DO
0050 SLSJC2=0.DO
0051 CLCJC2=0.DO
0052 SJS2CL=0.DO
0053 SLS2CJ=0.DO
C FIND THE VALUE OF THE TRIGONOMETRIC INTEGRALS FOR DIFFERENT I,J,L,K
0054 IF(L-J)4,3,4
0055 CALL CCC(CLCJ,L,J,C)
0056 CALL SSC(SLSJ,L,J,0)
0057 IF((L-J-2)*(L+J+2))5,6,5
0058 CALL CCC(CLCJC2,L,J,2)
0059 CALL SSC(SLSJC2,L,J,2)
0060 CALL SSC(SJS2CL,J,2,L)
0061 CALL SSC(SLS2CJ,L,2,J)
C IF ALL OF THE TRIGONOMETRIC INTEGRALS ARE ZERO, SET THE ENERGY
C INTEGRAL EQUAL TO ZERO FOR THESE PARTICULAR VALUES OF I,J,K,L.
0062 IF(CLCJ**2+SLSJ**2+SLSJC2**2+CLCJC2**2+SJS2CL**2+SLS2CJ**2)7,80,7
0063 CCONTINUE
C FCRM THE TERMS IN THE ENERGY INTEGRALS.
0064 CALL THEMS(I,J,K,L)
C SUM THE TERMS IN THE ENERGY INTEGRALS
0065 UBTI=UB1+UB2A+UB3B+UB4+UB3A+UB2B+UB5R+UB5A+UB5
0066 URTII=(1.DC-VNU)*(UB2B+UB3A+UB2A+UB3B)
0067 URTIII=2.DO*(1.DC-VNU)*(UB7-UB8A-UB8B+UB9)

```



```

0068 UETA=((UBTI-LBTII)*CLCJ+(UBTIIII)*SLSJ)/URD
0069 URTB=((UBTI-LBTII)*SLSJ+(UBTIIII)*CLCJ)/URD
0070 USRTA=(US7A-3.*US9A+2.*US8A)*SJS2CL+(US7B-3.*US9B+2.*US8B)*SJS2CL
0071 USRTB=-(US7A-3.*US9A+2.*US8A)*SLS2CJ-(US7B-3.*US9B+2.*US8B)*SJS2CJ
0072 USRA=(US1-US2)*CLCJ+(US1+3.*US3-4.*US2)*CLCJC2
0073 USRB=(US1-US2)*SLSJ+(US1+3.*US3-4.*US2)*SLSJC2
0074 USTB=(US4+US5)*CLCJ-(US4+3.*US6)*CLCJC2
0075 USTA=(US4+US5)*SLSJ-(US4+3.*US6)*SLSJC2
0076 USA=(USRA+USTB+USRTA)/USD
0077 USB=(USRB+USTB+USRTB)/USD
0078 TEST TO SEE IF SINE OR COSINE TERMS OR PRESENT AND SUM ACCORDINGLY
      IF (JOY-2) 1C,10,9
0079 C THE A MATRIX REPRESENTS THE STRETCH ENERGY INTEGRAL
0080 C THE B MATRIX REPRESENTS THE BENDING ENERGY INTEGRAL
0081 C
0082 C
0083 C
0084 C
0085 C
0086 C
0087 C
0088 C
0089 C
0090 C
0091 C
0092 C
0093 C
0094 C
0095 C
0096 C
0097 C
0098 C
0099 C
0100 C
0101 C
0102 C
0103 C
0104 C
0105 C
0106 C

      10 A(JROW,JCOL)=USA*THICK/2.DO
      B(JROW,JCOL)=UBTA*RIGID
      GC TO 8
      9 A(JROW,JCOL)=USB*THICK/2.DO
      B(JROW,JCOL)=UBTB*RIGID
      DOUBLE THE CIAGCNAL TERMS OF THE A AND B MATRICES
      8 IF(JCOL-JROW)12,13,12
      13 A(JROW,JCOL)=2.*A(JROW,JCOL)
      B(JROW,JCOL)=2.*B(JROW,JCOL)
0090 FORMAT(1H1,9HSYMMETRY=,2X,13,3X,2HN=,2X,13,3X,2HM=,2X,13,3X,6HTER
      IMS=,2X,13,3X,10HTHICKNESS=,G12.6/)
0088 12 CCNTINUE
      C IF A ,GO TO 800: CARD FOLLOWS, MATRIX ELEMENT PRINT IS SUPPRESSED
      GC TO 800
0089 887 IF(NEWPAG -50)23,25,25
0090 25 WRITE(6,990) JOY,N,M,INIT,THICK
0091 WRITE(6,910)
0092 910 FORMAT(1H,13H I J K L,5X,4HCLCJ,7X,4HSLSJ,7X,6HSLJC2,5X,6HC
      1LCJC2,5X,6HSLS2CJ,5X,6HSLS2CL,5X,7HSTRETCH,9X,4HBEND,/)
0093 23 WRITE(6,920)I,J,K,L,CLCJ,SLSJ,SLSJC2,CLCJC2,SJS2CL,A(JROW,J
      1COL),B(JROW,JCOL)
0094 920 FCRMAT(1X,4(I2,1X),2X,6(G10.3,1X),2X,2G15.8)
0095 800 NEWPAG=NEWPAG+1
      CCNTINUE
0096 WRITE(6,990) JOY,N,M,INIT,THICK
0097 SOLVE THE EIGENVALUE PROBLEM AX+R=0
0098 CALL JUGGLE(INIT)
0099 DO 225 I=1,INIT
      DO 225 J=1,INIT
0100 EIVR(I,J)=A(I,J)
0101 WRITE(6,990)JOY,N,M,INIT,THICK
0102 BETWEEN=ITIME(0)*.01
0103 TIME=BETWEEN-START
0104
0105
0106

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01C7 WRITE(6,995) TIME
01C8 NULL=0
01C9 DO 220 I=1,INIT
      DETERMINE WHICH EIGENVALUE S ARE POSITIVE
0110 IF (EIVU(I))220,220,221
0111 221 NULL=NULL+1
0112 TEMP=(NULL)=EIVU(I)
0113 DO 222 J=1,INIT
0114 EIVR(J,NULL)=EIVR(J,I)
0115 222 CONTINUE
0116 IF (NULL.EQ.C) GC TC 844
0117 NULL=1
0118 DO 223 NO=1,NULL
      PRINT THE MODE SHAPES FOR THE POSITIVE EIGENVALUES.
0119 CALL PICTUR (NC,50)
0120 CONTINUE
0121 BETWEEN=ITIME(C)*.01
0122 TIME=BETWEEN-START
0123 WRITE(6,995) TIME
0124 CONTINUE
0125 844 FORMAT(IX,'ELAPSED TIME = ',G13.5,'SECONDS')
0126 995 END

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0001 SUBROUTINE JUGGLE (LINK)
0002 THIS SUBROUTINE CONVERTS A MATRIX EIGENVALUE OF THE FORM (AX+B)Y=0
0003 TO THE FORM (IX+C)Z=0, WHERE A, B, C ARE COEFFICIENT MATRICES,
0004 X IS THE EIGENVALUE MATRIX, AND Y, Z ARE THE EIGENVECTOR MATRICES.
0005 SEE APPENDIX G. OF 'BUCKLING NEAR A HOLE IN AN INFINITE PLATE UNDER
0006 TENSION' BY C. A. PELLETT, NPGS MASTER'S THESIS.
0007 IMPLICIT REAL*8 (A-H,O-Z)
0008 COMMON/CHUNK/A(80,80),R(80,80),FIVR(80,80)
0009 DC 1 I=1,LINK
0010 DC 1 J=1,LINK
0011 FIVR(I,J)=A(I,J)
0012 A(I,J)=B(I,J)
0013 B(I,J)=EIVR(I,J)
0014 CALL JACVAT(LINK)
0015 DC 2 J=1,LINK
0016 DC 2 I=1,LINK
0017 SUM=0.00
0018 CC 3 JOKE=1,LINK
0019 SUM=EIVR(JOKE,I)*B(JOKE,J)+SUM
0020 A(I,J)=SUM
0021 DC 4 J=1,LINK
0022 DC 4 I=1,LINK
0023 SUM=0.00
0024 DC 5 JOKE=1,LINK
0025 SUM=A(I,JOKE)*EIVR(JOKE,J)+SUM
0026 B(I,J)=SUM
0027 DC 6 I=1,LINK
0028 TEMP A(I)=DSQRT(EIVU(I))
0029 EIVU(I)=1.00/DSQRT(EIVU(I))
0030 DC 7 I=1,LINK
0031 DC 7 J=1,LINK
0032 A(I,J)=B(I,J)*EIVU(I)*EIVU(J)
0033 B(I,J)=EIVR(I,J)
0034 CALL JACVAT(LINK)
0035 DC 8 I=1,LINK
0036 DC 8 J=1,LINK
0037 SUM=0.00
0038 DC 9 JOKE=1,LINK
0039 SUM=B(JOKE,I)*EIVR(JOKE,J)+SUM
0040 A(I,J)=SUM*TEMPA(I)
0041 DC 10 I=1,LINK
0042 EIVU(I)=-1.00/EIVU(I)
0043 CALL PRINN(4,LINK,0)
0044 RETURN
0045 END

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```

0001 SUBROUTINE CCNTR
0002 COMMON/CHUNK/A(150,130),DUMM(6100),DUM2(12800)
0003 N=150
0004 M=130

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0040

C THIS SUBROUTINE WILL GIVE A MAP OF THE NUMBERS CONTAINED IN THE ARRAY A
C THREE PARAMETERS MUST BE FURNISHED TO THE SUBROUTINE
C A IS THE NAME OF THE ARRAY TO BE MAPPED
C N IS THE NUMBER OF LINES IN T&E MAP
C M IS THE WIDTH OF THE LINE, THE MAXIMUM IS 115
C DIMENSION LINE(130),ISYMB(41),BV(21)
C DATA ISYMB/1F,1F,1HA,1H,1HB,1H,1HC,1H,1HD,1H,1HE
C 1,1H,1HF,1H,1HG,1H,1HH,1H,1HI,1H,1HJ,1H,1HK,1H,1HL,1H,1HM,1H,1HN,1H,1HO,1H,1HP,1H,1HQ,1H,1HR,1H,1HS,
C 21F,1HT,1H,1HU,1H,1HV,1H,1HW,1H,1HX,1H,1HY,1H,1HZ/
C DATA 1BLK/1F/
C AMIN=1.F+7C
C DC 6 I=1,N
C DC 5 J=1,M
C 5 AMIN=AMIN1(AMIN,A(I,J))
C 6 CCNTINUE
C DC 8 I=1,N
C DC 7 J=1,M
C 7 A(I,J)=A(I,J)-AMIN
C 8 CCNTINUE
C AMAX=-1.F+7C
C DC 11 I=1,N
C DC 10 J=1,M
C 10 AMAX=AMAX1(AMAX,A(I,J))
C 11 CCNTINUE
C SCF=4C./AMAX
C DC 21 I=1,N
C DC 20 J=1,M
C 20 A(I,J)=SCF*A(I,J)
C 21 CCNTINUE
C DC 30 I=1,21
C AI=DFLCAT(1-1)
C 30 BV(I)=I2.*AI/SCF)+AMIN
C WRITE(6,105C) BV(1),BV(1),BV(21)
C WRITE(6,105C) BV(2),BV(3),BV(4),BV(5),BV(6)
C WRITE(6,110C) BV(7),BV(8),BV(9),BV(10),BV(11)
C WRITE(6,120C) BV(12),BV(13),BV(14),BV(15),BV(16)
C WRITE(6,121C) BV(17),BV(18),BV(19),BV(20),BV(21)
C WRITE(6,105C)
C M1=M+1
C DC 50 K=1,N
C DC 40 J=1,M
C AJ=A(K,J)+1.5000C1

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0041 $AJ=AJ
0042 JJ=IFIX($AJ)
0043 LINE(JJ)=ISYMP(JJ)
0044 IF(M1-115) 41,41,46
0045 41 CONTINUE
0046 DO 45 I=M1,120
0047 LINE(I)=IBLK
0048 46 CONTINUE
0049 WRITE(6,2000) LINE
0050 50 CONTINUE
0051 DO 51 I=1,N
0052 DO 60 J=1,M
0053 A(I,J)=(A(I,J)/SCF)+AMIN
0054 61 CONTINUE
0055 WRITE(6,2500) NMAP
0056 2500 FORMAT(//,5X,4F MAP,I3)
0057 1050 FORMAT(//)
0058 1100 FORMAT(5X,'A=',G13.5,3X,'B=',G13.5,3X,'C=',G13.5,3X,'D=',G13.5,
0059 13X,'E=',G13.5)
0060 1200 FORMAT(5X,'F=',G13.5,3X,'G=',G13.5,3X,'H=',G13.5,3X,'I=',G13.5,
0061 13X,'J=',G13.5)
0062 1210 FORMAT(5X,'K=',G13.5,3X,'L=',G13.5,3X,'M=',G13.5,3X,'N=',G13.5,
0063 13X,'O=',G13.5)
0064 1220 FORMAT(5X,'P=',G13.5,3X,'Q=',G13.5,3X,'R=',G13.5,3X,'S=',G13.5,3X,'T=',G13.5,
0065 13X,'U=',G13.5)
0066 1230 FORMAT(5X,'V=',G13.5,3X,'W=',G13.5,3X,'X=',G13.5,3X,'Y=',G13.5,
0067 13X,'Z=',G13.5)
0068 1300 FORMAT(5X,3H 0=,G15.6,5X,7H RANGE(,G15.6,5X,G15.6,2H ))
0069 2000 FORMAT(1X,130A1)
0070 END

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C      THIS SUBROUTINE FORMS THE DEFLECTION MATRIX OF REPRESENTATIVE
C      POINTS ON THE PLATE FOR USE IN SUBROUTINE CONTR
SUBROUTINE PICTURE (NO,IRAD)
IMPLICIT REAL*8 (A-H,C-Z)
REAL*4 DATA,CUMM
COMMON/BAVE/ X(100),Y(100)
COMMON/TAVE/ START
COMMON/CHUNK/DATA(150,130),DUMM(6100),A(80,80)
COMMON/MUCK/EIVU(100),TEMPA(100)
COMMON/PASS/MCCUNT(100),JOY,N,M,INIT,THICK
RADIUS=IRAD
WRITE (6,990) JOY,N,M,INIT,THICK
WRITE (6,90) TEMPA(NO)
DC 125 NID=1,N
DC 125 MID=1,M
JRCW=(NID-1)*M+MID
EIVU(JROW)=A(JROW,N)
WRITE (6,91)NID,MCCUNT(MID),A(JROW,N)
CONTINUE
125 DC 210 IY=1,I3C
WY=IY
WY=WY-65.00
DC 210 IX=1,I5C
EX=IX/.800
QVARE=RADIUS/((CSQRT(EX*EX+WY*WY)))
WWW=0.00
IF(1.00-QVARE)21C,126,126
CONTINUE
126 THETA=ATAN2(WY,EX)
PARE=QVARE*OVARE
DC 213 NID=1,N
PARE=PAE*CVARE
N1=NID+1
N2=NID+2
N3=NID+3
DC 213 MID=1,M+MID
JROW=(NID-1)*M+MID
IF (JOY-2)214,214,215
214 WWW=WWW+(PARE*(NID*N1+OVARE*N2*(X(JROW)*N1+Y(JROW)*N3*OVARE)))
1000 TO 213
GC TO 213
215 WWW=WWW+(PARE*(NID*N1+OVARE*N2*(X(JROW)*N1+Y(JROW)*N3*OVARE)))
1000 TO 213
213 CONTINUE
210 DATA(IX,IY)=WWW
WRITE (6,92)

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0044 92 FORMAT(1X,'MAP OF THE RELATIVE CURVATURE IN THE RADIAL DIRECTION')
0045 CALL CCNTR
0046 90 FORMAT(1X,'CRITICAL BUCKLING STRESS = ',G12.5,' PSI',//1X,'COEFFICI
0047 91 FENTS OF THE SERIES',//)
0048 91 FORMAT(1X,'N=',I2,' M=',I2,' A(N,M) = ',G16.6)
0049 990 FORMAT(1H1,9HSYMMETRY=,2X,I3,3X,2HN=,2X,I3,3X,2HM=,2X,I3,3X,6HTER
0050 1MS=,2X,I3,3X,10HTHICKNESS=,G12.6/)
0051 RETURN
0052 444 FORMAT(1X,'ELAPSED TIME = ',G13.5,' SECONDS')
0053 END

```

```

0001 SUBROUTINE BCUND(N,M,X,Y)
0002 IMPLICIT REAL*8 (A-H,O-Z)
      C
      C THIS SUBROUTINE FINDS THE VALUES OF X AND Y FOR A GIVEN N AND M
      C SO THAT THE BOUNDARY CONDITIONS WILL BE SATISFIED.
0003 VNU=.3D0
0004 CMEN1=N*(N+1)-N*VNU-M**2*VNU
0005 CMEN2=(N+1)*(N+2)-(N+1)*VNU-M**2*VNU
0006 CMEN3=(N+2)*(N+3)-(N+2)*VNU-M**2*VNU
0007 SHE1=-N*(N+1)*(N+2)+N*(N+1)+N*(2-VNU)*M**2*N+(3-VNU
0008 SHE2=-{(N+1)*(N+2)*(N+3)+(N+1)*(N+2)+N*(2-VNU)*M**2*(N+1)+(3-VN
      1U)*M**2
0009 SHE3=-{(N+2)*(N+3)*(N+4)+(N+2)*(N+3)+(2-VNU)*M**2*(N+2)+(3-VN
      1U)*M**2
0010 X=(-CMEN1*SHE3+CMEN3*SHE1)/(CMEN2*SHE3-CMEN3*SHE2)
0011 Y=(-CMEN2*SHE1+CMEN1*SHE2)/(CMEN2*SHE3-CMEN3*SHE2)
0012 RETURN
0013 END

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0001      SUBROUTINE CCC(VALUE, IXX, IYY, IZZ)
      THIS SUBROUTINE FINDS THE VALUE OF THE INTEGRAL OF
      COSINE (IXX*THETA) COSINE (IYY*THETA) COSINE (IZZ*THETA)
      AS THETA RANGES FROM ZERO TO TWO PI
      IMPLICIT REAL*8 (A-H,O-Z)
      B1=0.00
      B2=0.000
      B3=0.000
      B4=0.000
      IF (IXX-IYY-IZZ) 1,2,1
2 1 IF (IXX+IYY+IZZ) 3,4,3
4 2 = .500
3 IF (IXX-IYY+IZZ) 5,6,5
6 3 = .500
5 IF (IXX+IYY-IZZ) 7,8,7
8 4 = .500
      VALUE=(B1+B2+B3+B4)*3.141592653589793238
      RETURN
      END
0002
0003
0004
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0014
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0016
0017
      C
      C
      C
      C
      C

```



```

0001 SUBROUTINE JACVAT (N)
0002 THIS SUBROUTINE SOLVES THE MATRIX EIGENVALUE PROBLEM (IX-A)Y=0.
0003 WHERE X IS THE EIGENVALUE MATRIX IN EIVU, Y IS THE EIGENVECTOR MATRIX
0004 IN EIVR, AND A IS THE COEFFICIENT MATRIX IN A
0005 IMPLICIT REAL*8(A-H,O-Z)
0006 COMMON/CHUNK/A(80,80),B(80,80),EIVR(80,80)
0007 COMMON/MUCK/EIVU(100),TEMPA(100)
0008 NOYES=1
0009 IF(N-1)20,23,21
0010 WRITE(6,22) N
0011 20 FORMAT(4H N=,I3,68H IS TOO SMALL. LIMIT IS 1. RETURN TO CALLING
0012 1 ROUTINE FROM JACVAT.)
0013 RETURN
0014 23 WRITE(6,24) A(1,1)
0015 24 FORMAT(24H IN JACVAT, MATRIX A =,D14.6)
0016 RETURN
0017 21 IF(N-160)1,1,3
0018 3 WRITE(6,2) N
0019 2 FORMAT(3H N=,I5,70H IS TOO LARGE. LIMIT IS 160. RETURN TO CALLING
0020 1 ROUTINE FROM JACVAT.)
0021 RETURN
0022 1 IF(NOYES)99,102,99
0023 99 CONTINUE
0024 DO 101 J=1,N
0025 DC 100 I=1,N
0026 EIVR(I,J)=0.000
0027 101 EIVR(J,J)=1.000
0028 CHECK SYMMETRY OF THE GIVEN MATRIX A. IF TWO SYMMETRIC ELEMENTS ARE
0029 NOT EQUAL, REPLACE EACH OF THE UNEQUAL ELEMENTS WITH THE MEAN OF THE
0030 TWO ELEMENTS, PRINT OUT A MESSAGE, AND CONTINUE WITH THE NEW MATRIX.
0031 ATCP=0.000
0032 DC 112 J=1,N
0033 DO 111 I=1,J
0034 IF(A(I,J)-A(J,I))90,103,90
0035 WRITE(6,106) N,N
0036 90 FORMAT(6,106) N,N
0037 106 JACVAT (A(I,1H,I3,1H,I3,3H),)
0038 WRITE(6,108) I,J,J,I
0039 108 FORMAT(3H A(I,1H,I3,1H,I3,10H) AND A(I,1H,I3,1H,I3,54H) WERE UNEQUAL,
0040 SO THEY WERE REPLACED WITH THEIR MEAN)
0041 100 CONTINUE
0042 A(I,J)=.5*(A(I,J)+A(J,I))
0043 A(J,I)=A(I,J)

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0033      103 CUNTINUE
C
C      FIND THE ABSOLUTELY LARGEST ELEMENT OF A
C
0034      IF(ATCP-DABS(A(I,J)))104,111,111
0035      ATOP=DABS(A(I,J))
0036      104 CCONTINUE
0037      111 EIVU(J)=A(J,J)
0038      IF(ATOP)109,109,113
0039      IF(ATOP)109,109,113
0040      WRITE (6,110)
0041      FORMAT(26H IN JACVAT, MATRIX A = 0 )
      RETURN
C
C      CALCULATE THE STOPPING CRITERION -- DSTOP
C
0042      113 AVGF=DFLOAT(N*(N-1))*0.55
0043      D=0.0D0
0044      DO 114 JJ=2,N
0045      DO 114 II=2,JJ
0046      S=A(II-1,JJ)/ATOP
0047      D=S*S+D
0048      114 DSTOP=(1.D-12)*D
C
C      CALCULATE THE THRESHOLD, THRSH
C
0049      THRSH = DSQRT(D/AVGF)*ATOP
C
C      START A SWEEP
C
0050      115 IFLAG=0
0051      DO 130 JCCL=2,N
0052      JCCL1=JCCL-1
0053      DO 130 IROW=1,JCCL1
0054      AIJ=A(IROW,JCCL)
C
C      COMPARE THE OFF-DIAGONAL ELEMENT WITH THRSH
C
0055      IF(DABS(AIJ)-THRSH)130,130,117
0056      117 AIJ=A(IROW,IROW)
0057      AIJ=A(JCCL,JCCL)
0058      S=AIJ-AIJ
C
C      CHECK TO SEE IF THE CHOSEN ROTATION IS LESS THAN THE ROUNDING ERROR.
C      IF SO , THEN DO NOT ROTATE.
C
0059      IF(DABS(AIJ)-1.D-17*DABS(S))130,130,118
0060      118 IFLAG=1
C

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0061      IF THE ROTATION IS VERY CLOSE TO 45 DEGREES, SET SIN AND COS
0062      TO 1/(RCCT 2).
0063
0064      IF(1.0-10*DABS(AIJ)-DABS(S))116,119,119
119      S=DSQRT(.5DC)
      C=S
      GO TO 120

0065      CALCULATION OF SIN AND COS FOR ROTATION THAT IS NOT VERY CLOSE
0066      TO 45 DEGREES
      T=AIJ/S
      S=C.25/DSQRT(C.25+T*T)
      COS = C , SIN= S

0067      C=DSQRT(0.5+S)
0068      S=2.*T*S/C

      CALCULATION OF THE NEW ELEMENTS OF MATRIX A

0069      DO 121 I=1,IRCW
0070      T=A(I,IRCW)
0071      U=A(I,JCOL)
0072      A(I,IRCW)=C*T-S*U
0073      A(I,JCOL)=S*T+C*U
0074      I2=IROW+2
0075      IF(I2-JCOL)127,127,123
127      CONTINUE
      DO 122 I=12,JCOL
      T=A(I-1,JCOL)
      U=A(IROW,I-1)
      A(I-1,JCOL)=S*U+C*T
      A(IROW,I-1)=C*U-S*T
122      A(JCOL,JCOL)=S*AIJ+C*AJJ
123      A(IRCW,IRCW)=C*A(IROW,IRCW)-S*(C*AIJ-S*AJJ)
      DO 124 J=JCOL,N
      T=A(IRCW,J)
      U=A(JCOL,J)
      A(IRCW,J)=C*T-S*U
      A(JCOL,J)=S*T+C*U
124

      ROTATION COMPLETED.
      SEE IF EIGENVECTORS ARE WANTED BY USER

0089      IF(NOVES)131,126,131
0090      CONTINUE
0091      DO 125 I=1,N

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0092      I=EIVR(I,I,KLW)
0093      EIVR(I,IRGW)=C*T-EIVR(I,JCOL)*S
0094      EIVR(I,JCOL)=S*T+EIVR(I,JCOL)*C
      C
      C      CALCULATE THE NEW NORM D AND COMPARE WITH DSTOP
      C
0095      126 CONTINUE
0096      S=AIJ/ATCP
0097      D=D-S*S
0098      IF(C-DSTOP)126C,129,129
      C
      C      RECALCULATE DSTOP AND THRSH TO DISCARD ROUNDING ERRORS
      C
0099      126C C=C.0D0
0100      DC 128 JJ=2,N
0101      DC 128 II=2,JJ
0102      S=A(II-1,JJ)/ATOP
0103      C=S*S+D
0104      DSTOP=(1.D-12)*D
0105      129 THRSH=DSQRT(C/AVGF)*ATOP
0106      130 CONTINUE
0107      IF(IFLAG)115,134,115
      C
      C      ENDING ROUTINE -- RESET MATRIX A AND PLACE EIGENVALUES IN EIVU
      C
0108      134 T=A(1,1)
0109      A(1,1)=EIVU(1)
0110      EIVU(1)=T
0111      DC 132 J=2,N
0112      T=A(J,J)
0113      A(J,J)=EIVU(J)
0114      EIVU(J)=T
0115      DC 132 I=2,J
0116      132 A(I-1,J)=A(J,I-1)
0117      133 RETURN
0118      END

```

0001	C	SUBROUTINE TERMS(I,J,K,L)
0002	C	THIS SUBROUTINE FINDS THE INTEGRATED VALUE OF THE TERMS OF THE
0003		STRETCH AND PENDING INTEGRALS FOR PLATE ENERGY
0004		IMPLICIT REAL*8(A-H,O-Z)
0005		COMMON/PAST/JRCW,JCCL
		COMMON/BANE/XX(100),YY(100)
		COMMON/NUMBER/US1,US2,US3,US4,US5,US6,US7A,US7B,US8A,US8B,US9A,
		US9B,UB1,UB2A,UB2B,UB3A,UB3B,UB4,UB5A,UB5B,UB7,UB8A,UB8B,UB9,UB6,
		2 UB0,USD
0006		DIMENSION BUB(9)
0007		X=XX(JCCL)
0008		XSTAR=XX(JRCW)
0009		Y=YY(JCCL)
0010		YSTAR=YY(JRCW)
0011		US1=0.00
0012		US2=0.00
0013		US3=0.00
0014		US4=0.00
0015		US5=0.00
0016		US6=0.00
0017		US7A=0.00
0018		US7B=0.00
0019		US8A=0.00
0020		US8B=0.00
0021		US9A=0.00
0022		US9B=0.00
0023		UB1=0.00
0024		UB2A=0.00
0025		UB2B=0.00
0026		UB3A=0.00
0027		UB3B=0.00
0028		UB4=0.00
0029		UB5A=0.00
0030		UB5B=0.00
0031		UB6=C.00
0032		UB7=0.00
0033		UB8A=0.00
0034		UB8B=0.00
0035		UB9=0.00
0036		TOR=I+K
0037		TCR1=I+K+1
0038		TCR2=I+K+2
0039		TCR3=I+K+3
0040		TCR4=I+K+4
0041		TCR5=I+K+5
0042		TCR6=I+K+6
0043		TCR7=I+K+7


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0044 TOR9=I+K+8
0045 DENOM=TOR1*TCR2*TOR3*TOR4*TOR5*TOR6*TOR7*TOR8
0046 DENOM1=TOR*TCR2*TOR3*TOR4*TOR5*TOR6*TOR7*TOR8
0047 DENOM2=TCR*TCR1*TOR3*TOR4*TOR5*TOR6*TOR7*TOR8
0048 DENOM3=TCR*TCR1*TOR2*TOR3*TOR5*TOR6*TOR7*TOR8
0049 DENOM4=TCR*TCR1*TOR2*TOR3*TOR4*TOR6*TOR7*TOR8
0050 DENOM5=TCR*TCR1*TOR2*TOR3*TOR4*TOR5*TOR7*TOR8
0051 DENOM6=TCR*TCR1*TOR2*TOR3*TOR4*TOR5*TOR6*TOR8
0052 DENOM7=TCR*TCR1*TOR2*TOR3*TOR4*TOR5*TOR6*TOR7
0053 DENOM8=TCR*TCR1*TOR2*TOR3*TOR4*TOR5*TOR6*TOR7
0054 USD=TOR#DENOM
0055 UBD=TOR#DENOM
0056 RUB(1)=I*K*(I+1)*(K+1)*DENOM2
0057 RUB(2)=(I+1)*(K+1)*(I+2)*(K+2)*X*STAR#DENOM4
0058 RUB(3)=(I+2)*(K+2)*(I+3)*(K+3)*Y*STAR#DENOM6
0059 RUB(4)=K*(K+1)*(I+1)*(I+2)*X*STAR#DENOM3
0060 RUB(5)=I*(I+1)*(K+1)*(K+2)*X*STAR#DENOM4
0061 RUB(6)=K*(K+1)*(I+2)*(I+3)*X*STAR#DENOM5
0062 RUB(7)=(I+1)*(I+2)*(K+2)*(K+3)*Y*STAR#DENOM4
0063 RUB(8)=I*(I+1)*(K+2)*(K+3)*Y*STAR#DENOM5
0064 RUB(9)=(K+1)*(K+2)*(I+2)*(I+3)*X*STAR#DENOM5
0065 UBI=0
0066 DOL KK=1,9
0067 1 UBI=8UB(KK)+UB1
0068 RUB(1)=-I*K*(K+1)*DENOM2
0069 RUB(2)=-I*(K+1)*(K+2)*Y*STAR#DENOM3
0070 RUB(3)=-I*(K+2)*(K+3)*Y*STAR#DENOM4
0071 RUB(4)=-I*(I+1)*K*(K+1)*X*STAR#DENOM3
0072 RUB(5)=-I*(I+1)*(K+1)*(K+2)*X*STAR#DENOM4
0073 RUB(6)=-I*(I+1)*(K+2)*(K+3)*X*STAR#DENOM5
0074 RUB(7)=-I*(I+2)*K*(K+1)*Y*STAR#DENOM4
0075 RUB(8)=-I*(I+2)*(K+1)*(K+2)*Y*STAR#DENOM5
0076 RUB(9)=-I*(I+2)*(K+2)*(K+3)*Y*STAR#DENOM6
0077 DG 2 KK=1,9
0078 2 UB2A=UB2A+8UB(KK)
0079 RUB(1)=-K*I*(I+1)*DENOM2
0080 RUB(2)=-K*(I+1)*(I+2)*X*STAR#DENOM3
0081 RUB(3)=-K*(I+2)*(I+3)*Y*STAR#DENOM4
0082 RUB(4)=-K*(I+1)*I*(I+1)*X*STAR#DENOM3
0083 RUB(5)=-K*(I+1)*(I+1)*(I+2)*X*STAR#DENOM4
0084 RUB(6)=-K*(I+1)*(I+2)*(I+3)*X*STAR#DENOM5
0085 RUB(7)=-K*(I+2)*I*(I+1)*Y*STAR#DENOM4
0086 RUB(8)=-K*(I+2)*(I+1)*(I+2)*Y*STAR#DENOM5
0087 RUB(9)=-K*(I+2)*(I+3)*Y*STAR#DENOM6
0088 DG 3 KK=1,9
0089 3 UB2B=UB2B+8UB(KK)
0090 RUB(1)=-I*(I+1)*L**2#DENOM2
0091 RUB(2)=-I*(I+1)*L**2#DENOM3

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0092  BUH(3)=-I*(I+1)*L**2*Y*DENOM4
0093  BUH(4)=-((I+1))*((I+2))*XSTAR*L**2**DENOM3
0094  BUH(5)=-((I+1))*((I+2))*XSTAR*X*L**2**DENOM4
0095  BUH(6)=-((I+1))*((I+2))*XSTAR*Y*L**2**DENOM5
0096  BUH(7)=-((I+2))*((I+3))*YSTAR*L**2**DENOM4
0097  BUH(8)=-((I+2))*((I+3))*YSTAR*L**2**X**DENOM5
0098  BUH(9)=-((I+2))*((I+3))*YSTAR*L**2**Y**DENOM6
0099  DO 4 KK=1,9
0100  UB3A=UB3A+BUH(KK)
0101  BUH(1)=-K*((K+1))*J**2**DENOM2
0102  BUH(2)=-K*((K+1))*J**2**XSTAR**DENOM3
0103  BUH(3)=-K*((K+1))*J**2**YSTAR**DENOM4
0104  BUH(4)=-((K+1))*((K+2))*J**2**X**DENOM3
0105  BUH(5)=-((K+1))*((K+2))*J**2**X**XSTAR**DENOM4
0106  BUH(6)=-((K+1))*((K+2))*J**2**X**YSTAR**DENOM5
0107  BUH(7)=-((K+2))*((K+3))*Y**J**2**DENOM4
0108  BUH(8)=-((K+2))*((K+3))*Y**J**2**XSTAR**DENOM5
0109  BUH(9)=-((K+2))*((K+3))*Y**J**2**YSTAR**DENOM6
0110  DO 5 KK=1,9
0111  UB3B=UB3B+BUH(KK)
0112  BUH(1)=I*K**DENOM2
0113  BUH(2)=K*((I+1))*XSTAR**DENOM3
0114  BUH(3)=K*((I+2))*YSTAR**DENOM4
0115  BUH(4)=(K+1)*I*X**DENOM3
0116  BUH(5)=(K+1))*((I+1))*X**XSTAR**DENOM4
0117  BUH(6)=(K+1))*((I+2))*X**YSTAR**DENOM5
0118  BUH(7)=(K+2))*((I+1))*Y**DENOM4
0119  BUH(8)=(K+2))*((I+1))*XSTAR*Y**DENOM5
0120  BUH(9)=(K+2))*((I+2))*Y**YSTAR**DENOM6
0121  DO 6 KK=1,9
0122  UB4=UB4+BUH(KK)
0123  BUH(1)=J**2**K**DENOM2
0124  BUH(2)=J**2**((K+1))*X**DENOM3
0125  BUH(3)=J**2**((K+2))*Y**DENOM4
0126  BUH(4)=J**2**K**XSTAR**DENOM3
0127  BUH(5)=J**2**((K+1))*XSTAR*X**DENOM4
0128  BUH(6)=J**2**((K+2))*XSTAR*Y**DENOM5
0129  BUH(7)=J**2**K**YSTAR**DENOM4
0130  BUH(8)=J**2**((K+1))*YSTAR*X**DENOM5
0131  BUH(9)=J**2**((K+2))*YSTAR*Y**DENOM6
0132  DO 7 KK=1,9
0133  UB5A=UB5A+BUH(KK)
0134  BUH(1)=L**2**I**DENOM2
0135  BUH(2)=L**2**((I+1))*XSTAR**DENOM3
0136  BUH(3)=L**2**((I+2))*YSTAR**DENOM4
0137  BUH(4)=L**2**I**X**DENOM3
0138  BUH(5)=L**2**((I+1))*XSTAR*X**DENOM4
0139  BUH(6)=L**2**((I+2))*X**YSTAR**DENOM5

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BUB(7)=L**2*I*Y*DENOM4
BUB(8)=L**2*(I+1)*XSTAR*Y*DENOM5
BUB(9)=L**2*(I+2)*YSTAR*Y*DENOM6
DO 8 KK=1,9
UR58=UB58+BUB(KK)
BUB(1)=J**2*L**2*DENOM2
BUB(2)=J**2*L**2*XSTAR*DENOM3
BUB(3)=J**2*L**2*YSTAR*DENOM4
BUB(4)=J**2*L**2*X*DENOM3
BUB(5)=J**2*L**2*X*XSTAR*DENOM4
BUB(6)=J**2*L**2*X*YSTAR*DENOM5
BUB(7)=J**2*L**2*Y*DENOM4
BUB(8)=J**2*L**2*Y*YSTAR*DENOM5
BUB(9)=J**2*L**2*Y*YSTAR*DENOM6
DC 9 KK=1,9
UB6=UB6+BUB(KK)
BUB(1)=I*J*K*L*DENOM2
BUB(2)=I*J*(K+1)*L*X*DENOM3
BUB(3)=I*J*(K+2)*L*Y*DENOM4
BUB(4)=(I+1)*J*K*L*XSTAR*DENOM3
BUB(5)=(I+1)*J*(K+1)*L*XSTAR*X*DENOM4
BUB(6)=(I+1)*J*(K+2)*L*XSTAR*Y*DENOM5
BUB(7)=(I+2)*J*K*L*YSTAR*DENOM4
BUB(8)=(I+2)*J*(K+1)*L*YSTAR*X*DENOM5
BUB(9)=(I+2)*J*(K+2)*L*YSTAR*Y*DENOM6
DO 10 KK=1,9
UB7=UB7+BUB(KK)
BUB(1)=-J*K*DENOM2
BUB(2)=-J*(K+1)*X*DENOM3
BUB(3)=-J*(K+2)*Y*DENOM4
BUB(4)=-J*K*XSTAR*DENOM3
BUB(5)=-J*(K+1)*XSTAR*X*DENOM4
BUB(6)=-J*(K+2)*XSTAR*Y*DENOM5
BUB(7)=-J*K*YSTAR*DENOM4
BUB(8)=-J*(K+1)*YSTAR*X*DENOM5
BUB(9)=-J*(K+2)*YSTAR*Y*DENOM6
DC 11 KK=1,9
UB8A=UB8A+BUB(KK)
BUB(1)=-L*I*DENOM2
BUB(2)=-L*(I+1)*XSTAR*DENOM3
BUB(3)=-L*(I+2)*YSTAR*DENOM4
BUB(4)=-L*I*X*DENOM3
BUB(5)=-L*(I+1)*X*XSTAR*DENOM4
BUB(6)=-L*(I+2)*X*YSTAR*DENOM5
BUB(7)=-L*I*Y*DENOM4
BUB(8)=-L*(I+1)*Y*XSTAR*DENOM5
BUB(9)=-L*(I+2)*Y*YSTAR*DENOM6
DO 12 KK=1,9

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0236	BUB(4)=J*L**X*DENCM1	
0237	BUB(5)=J*L**X*XSTAR*DENCM2	
0238	BUB(6)=J*L**X*YSTAR*DENCM3	
0239	BUB(7)=J*L**Y*DENCM2	
0240	BUB(8)=J*L**Y*XSTAR*DENCM3	
0241	BUB(9)=J*L**Y*YSTAR*DENCM4	
0242	DO 17 KK=1,9	
0243	US4=US4+BUB(KK)	
0244	BUB(1)=J*L*DENCM2	
0245	BUB(2)=J*L*XSTAR*DENCM3	
0246	BUB(3)=J*L*YSTAR*DENCM4	
0247	BUB(4)=J*L*X*XSTAR*DENCM3	
0248	BUB(5)=J*L*X*XSTAR*DENCM4	
0249	BUB(6)=J*L*X*YSTAR*DENCM5	
0250	BUB(7)=J*L*Y*DENCM4	
0251	BUB(8)=J*L*Y*XSTAR*DENCM5	
0252	BUB(9)=J*L*Y*YSTAR*DENCM6	
0253	DO 18 KK=1,9	
0254	US5=US5+BUB(KK)	
0255	BUB(1)=J*L*DENCM4	
0256	BUB(2)=J*L*XSTAR*DENCM5	
0257	BUB(3)=J*L*YSTAR*DENCM6	
0258	BUB(4)=J*L*X*XSTAR*DENCM5	
0259	BUB(5)=J*L*X*XSTAR*DENCM6	
0260	BUB(6)=J*L*X*YSTAR*DENCM7	
0261	BUB(7)=J*L*Y*DENCM6	
0262	BUB(8)=J*L*Y*XSTAR*DENCM7	
0263	BUB(9)=J*L*Y*YSTAR*DENCM8	
0264	DO 19 KK=1,9	
0265	US6=US6+BUB(KK)	
0266	BUB(1)=J*K*DENOM	
0267	BUB(2)=J*(K+1)*X*DENCM1	
0268	BUB(3)=J*(K+2)*Y*DENCM2	
0269	BUB(4)=J*K*XSTAR*DENOM1	
0270	BUB(5)=J*(K+1)*XSTAR*X*DENOM2	
0271	BUB(6)=J*(K+2)*XSTAR*Y*DENOM3	
0272	BUB(7)=J*K*YSTAR*DENOM2	
0273	BUB(8)=J*(K+1)*YSTAR*X*DENOM3	
0274	BUB(9)=J*(K+2)*YSTAR*Y*DENOM4	
0275	DO 20 KK=1,9	
0276	US7A=US7A+BUB(KK)	
0277	BUB(1)=-L*I*DENOM	
0278	BUB(2)=-L*(I+1)*XSTAR*DENOM1	
0279	BUB(3)=-L*(I+2)*YSTAR*DENOM2	
0280	BUB(4)=-L*I*X*DENOM1	
0281	BUB(5)=-L*(I+1)*XSTAR*X*DENOM2	
0282	BUB(6)=-L*(I+2)*YSTAR*Y*DENOM3	
0283	BUB(7)=-L*I*Y*DENOM2	


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0284 BUB(8)=-L*(I+1)*Y*STAR*DENOM3
0285 BUB(9)=-L*(I+2)*Y*STAR*Y*DENOM4
0286 DO 21 KK=1,9
0287 21 US7B=US7B+BUB(KK)
0288 BUB(1)=-J*K*DENOM2
0289 BUB(2)=-J*(K+1)*X*DENOM3
0290 BUB(3)=-J*(K+2)*Y*DENOM4
0291 BUB(4)=-J*K*STAR*DENOM3
0292 BUB(5)=-J*(K+1)*X*STAR*X*DENOM4
0293 BUB(6)=-J*(K+2)*X*STAR*Y*DENOM5
0294 BUB(7)=-J*K*Y*STAR*DENOM4
0295 BUB(8)=-J*(K+1)*Y*STAR*X*DENOM5
0296 BUB(9)=-J*(K+2)*Y*STAR*Y*DENOM6
0297 DO 22 KK=1,9
0298 22 US8A=US8A+BUB(KK)
0299 BUB(1)=-L*I*DENOM2
0300 BUB(2)=-L*(I+1)*X*STAR*DENOM3
0301 BUB(3)=-L*(I+2)*Y*STAR*DENOM4
0302 BUB(4)=-L*I*X*DENOM3
0303 BUB(5)=-L*(I+1)*X*STAR*X*DENOM4
0304 BUB(6)=-L*(I+2)*X*Y*STAR*DENOM5
0305 BUB(7)=-L*I*Y*DENOM4
0306 BUB(8)=-L*(I+1)*Y*X*STAR*DENOM5
0307 BUB(9)=-L*(I+2)*Y*STAR*Y*DENOM6
0308 DO 23 KK=1,9
0309 23 US8B=US8B+BUB(KK)
0310 BUB(1)=-J*K*DENOM4
0311 BUB(2)=-J*(K+1)*X*DENOM5
0312 BUB(3)=-J*(K+2)*Y*DENOM6
0313 BUB(4)=-J*K*STAR*DENOM5
0314 BUB(5)=-J*(K+1)*X*STAR*X*DENOM6
0315 BUB(6)=-J*(K+2)*X*STAR*Y*DENOM7
0316 BUB(7)=-J*K*Y*STAR*DENOM6
0317 BUB(8)=-J*(K+1)*Y*STAR*X*DENOM7
0318 BUB(9)=-J*(K+2)*Y*STAR*Y*DENOM8
0319 DO 24 KK=1,9
0320 24 US9A=US9A+BUB(KK)
0321 BUB(1)=-L*I*DENOM4
0322 BUB(2)=-L*(I+1)*X*STAR*DENOM5
0323 BUB(3)=-L*(I+2)*Y*STAR*DENOM6
0324 BUB(4)=-L*I*X*DENOM5
0325 BUB(5)=-L*(I+1)*X*STAR*X*DENOM6
0326 BUB(6)=-L*(I+2)*X*Y*STAR*DENOM7
0327 BUB(7)=-L*I*Y*DENOM6
0328 BUB(8)=-L*(I+1)*Y*X*STAR*DENOM7
0329 BUB(9)=-L*(I+2)*Y*STAR*Y*DENOM8
0330 DO 25 KK=1,9
0331 25 US9B=US9B+BUB(KK)

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RETURN
END

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00061 SUBROUTINE PRINN (N,M,NPAGE)
      THIS SUBROUTINE PRINTS THE MATRIX A,B,OR EIVR, DEPENDING ON
      WHETHER N IN THE CALL ARGUMENT IS 1,2,OR 3. IF NPAGE EQUALS ZERO
      NC PAGE HEADING WILL BE PRODUCED. IF NPAGE EQUALS 1 A NEW PAGE
      WILL BE STARTED. THIS SUBROUTINE FINDS UTILITY IN DEBUGGING
      WHEN MATRIX PRINTOUT IS NEEDED.
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION B(1,1)
      COMMON/MUCK/EIVU(100),TEMPA(100)
      COMMON/CHUNK/A(80,80),B(80,80),EIVR(80,80)
      IF (NPAGE.EQ.1) WRITE (6,99)
      99 FORMAT ('1,')
      100 GO TO (1,2,3,14,15),N
      101 WRITE(6,91)
      91 FORMAT('1, MATRIX IN A'//)
      102 GO TO 4
      103 WRITE (6,92)
      92 FORMAT('1, MATRIX IN B'//)
      104 GO TO 4
      105 WRITE (6,93)
      93 FORMAT('1, MATRIX IN EIVR'//)
      106 GO TO 4
      107 WRITE(6,84)
      108 FORMAT('1, MATRIX IN EIVU'//)
      109 WRITE (6,88)((J,EIVU(J)),J=1,M)
      110 GO TO 100
      111 WRITE (6,85)
      112 FORMAT('1, MATRIX IN TEMPA'//)
      113 WRITE (6,88)((J,TEMPA(J)),J=1,M)
      114 GO TO 100
      115 DO 8 I=1,M
      116 GO TO (5,6,7),N
      117 WRITE (6,94)(A(I,J),J=1,M)
      118 WRITE(6,95)
      119 GO TO 8
      120 WRITE (6,94)(B(I,J),J=1,M)
      121 WRITE(6,95)
      122 GO TO 8
      123 WRITE (6,94)(EIVR(I,J),J=1,M)
      124 WRITE(6,95)
      125 CCNTINUE
      126 CCNTINUE
      127 RETURN
      128 FORMAT(1X,'EIG('12,')= ',G15.7)
      129 FORMAT (1X,10G12.5)

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0041 95 FCRMAT(IX)
0042 END

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13. ABSTRACT

A study is made of the circumstances under which there will be incipient buckling in an infinite plate containing a circular hole and subjected to uniform uniaxial tension at a great distance from the hole. This buckling is a consequence of the fact that near the hole and in line with the direction of the applied tensile loading, there are areas in which both principal membrane stresses are compressive. Values of critical tensile stress are given, together with expressions for the corresponding lateral deflection patterns. The results are in the form

$$S_{cr} = KE(t/a)^2$$

where E is Young's modulus of elasticity, t is plate thickness, a is a hole radius, and K is a buckling coefficient having the following values for the first four modes: 7.65, 7.80, 11.7, 12.3. A comparison is made with some experimental determinations of Danis for plates of finite width.

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KEY WORDS

LINK A

LINK B

LINK C

ROLE

WT

ROLE

WT

ROLE

WT

Plate Buckling
Energy Method
Rayleigh Ritz Method
Infinite Plate
Stress Concentration
Plate with a Hole
Eigenvalue Problem



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